SETS AND ELEMENTS

A set is a collection of objects called the elements or members of the set. The ordering of the elements is not important and repetition of elements is ignored, for example \{1, 3, 1, 2, 2, 1\} = \{1, 2, 3\}.

One usually uses capital letters, A, B, X, Y, . . . , to denote sets, and lowercase letters, a, b, x, y, . . . , to denote elements of sets.

Below you'll see just a sampling of items that could be considered as sets:

- The items in a store
- The English alphabet
- Even numbers

A set could have as many entries as you would like. It could have one entry, 10 entries, 15 entries, infinite number of entries, or even have no entries at all!

For example, in the above list the English alphabet would have 26 entries, while the set of even numbers would have an infinite number of entries.

Each entry in a set is known as an element or member.

Sets are written using curly brackets "\{" and "\}" , with their elements listed in between.

For example the English alphabet could be written as:
\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}
and even numbers could be \( \{0, 2, 4, 6, 8, 10, \ldots \} \) (Note: the dots at the end indicating that the set goes on infinitely)

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**Principles:**

- \( \in \) belong to
- \( \notin \) not belong to
- \( \subseteq \) subset
- \( \subset \) proper subset, for example, \( \{a, b\} \) is a proper subset of \( \{a, b, c\} \), but \( \{a, b, c\} \) is not a proper subset of \( \{a, b, c\} \).
- \( \not\subseteq \) not subset

So we could replace the statement "ais belong to the alphabet" with \( a \in \{\text{alphabet}\} \) and replace the statement "3 is not belong to the set of even numbers" with \( 3 \notin \{\text{Even numbers}\} \).

Now if we named our sets we could go even further.

Give the set consisting of the alphabet \( \text{alphabet} \) thename \( A \), and give the set consisting of even numbers \( \text{even numbers} \) thename \( E \).

We could now write

\[ a \in A \]

and

\[ 3 \notin E. \]

**Problem**

Let \( A = \{2, 3, 4, 5\} \) and \( C = \{1, 2, 3, \ldots, 8, 9\} \), Showthat \( A \) is a proper subset of \( C \).

**Answer**

Each element of \( A \) belongs to \( C \) so \( A \subseteq C \). On the other hand, \( 1 \in C \) but \( 1 \notin A \).

Hence \( A \neq C \). Therefore \( A \) is a proper subset of \( C \).

There are three ways to specify a particular set:

1) By list its members separated by commas and contained in braces \( \{ \} \), (if it is possible), for example, \( A = \{a, e, i, o, u\} \)

2) By stating those properties which characterize the elements in the set, for example, \( A = \{x : x \text{ is a letter in the English alphabet, } x \text{ is a vowel}\} \)

3) Venn diagram: (A graphical representation of sets).
Example (1)

\[ A = \{ x : x \text{ is a letter in the English alphabet, } x \text{ is a vowel} \} \]

\[ e \in A (e \text{ is belong to } A) \]

\[ f \notin A (f \text{ is not belong to } A) \]

Example (2)

\[ X \text{ is the set } \{1, 3, 5, 7, 9\} \]

\[ 3 \in X \text{ and } 4 \notin X \]

Example (3)

Let \[ E = \{ x | x^2 - 3x + 2 = 0 \} \rightarrow (x - 2)(x - 1) = 0 \rightarrow x = 2 \text{ & } x = 1 \]

\[ E = \{ 2, 1 \}, \text{ and } 2 \notin \emptyset \]

**Universal set, empty set:**

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set. For example, in human population studies, the universal set consists of all the people in the world. We will let the symbol \( U \) denote the universal set. The set with no elements is called the empty set or null set and is denoted by \( \emptyset \) or \( \{ \} \).

**Subsets:**

Every element in a set \( A \) is also an element of set \( B \), then \( A \) is called a subset of \( B \). We also say that \( B \) contains \( A \). This relationship is written:

\[ A \subseteq B \text{ or } B \supseteq A \]

If \( A \) is not a subset of \( B \), i.e. if at least one element of \( A \) does not belong to \( B \), we write \( A \nsubseteq B \).

**Example 4:**

Consider these sets.

\[ A = \{ 1, 3, 4, 5, 8, 9 \} \quad B = \{ 1, 2, 3, 5, 7 \} \quad \text{and } \quad C = \{ 1, 5 \} \]

Then \( C \subseteq A \) and \( C \subseteq B \) since 1 and 5, the elements of \( C \), are also members of \( A \) and \( B \). But \( B \nsubseteq A \) since 7, one of the elements of \( B \), does not belong to \( A \). Furthermore, since the elements of \( A, B \) and \( C \) must also belong to the universal set \( U \), we have that \( U \) must at least be the set \( \{ 1, 2, 3, 4, 5, 7, 8, 9 \} \).
Thenotion of subsets is graphically illustrated below:

A is entirely within B so \( A \subset B \).
A and B are disjoint or \( A \cap B = \emptyset \) so we could write \( A \notin B \) and \( B \notin A \).

**Set of numbers:**

Several sets are used so often, they are given special symbols.

- **\( \mathbb{N} \)** = the set of *natural numbers* or positive integers
  \[ \mathbb{N} = \{0, 1, 2, 3, \ldots\} \]

- **\( \mathbb{Z} \)** = the set of all integers: \( \ldots, -2, -1, 0, 1, 2, \ldots \)
  \[ \mathbb{Z} = \mathbb{N} \cup \{\ldots, -2, -1\} \]

- **\( \mathbb{Q} \)** = the set of rational numbers
  \[ \mathbb{Q} = \mathbb{Z} \cup \{\ldots, -1/3, -1/2, 1/2, 1/3, \ldots, 2/3, 2/5, \ldots\} \]
  Where \( \mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\} \)

- **\( \mathbb{R} \)** = the set of real numbers

- **\( \mathbb{C} \)** = the set of complex numbers
  \[ \mathbb{C} = \mathbb{R} \cup \{i, 1 + i, 1 - i, \sqrt{2} + \pi i, \ldots\} \]
  Where \( \mathbb{C} = \{x + iy : x, y \in \mathbb{R}; i = \sqrt{-1}\} \) Observe that \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \).

**Theorem 1:**

For any set A, B, C:
1- $\emptyset \subset A \subset U$.

2- $A \subset A$.

3- If $A \subset B$ and $B \subset C$, then $A \subset C$.

4- $A = B$ if and only if $A \subset B$ and $B \subset A$.

**Set operations:**

1) **UNION:**

The union of two sets $A$ and $B$, denoted by $A \cup B$, is the set of all elements which belong to $A$ or $B$;

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

**Example**

$A = \{1, 2, 3, 4, 5\}$ $B = \{5, 7, 9, 11, 13\}$

$A \cup B = \{1, 2, 3, 4, 5, 7, 9, 11, 13\}$

2) **INTERSECTION**

The intersection of two sets $A$ and $B$, denoted by $A \cap B$, is the set of elements which belong to both $A$ and $B$;

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$ 

**Example 1**

$A = \{1, 3, 5, 7, 9\}$ $B = \{2, 3, 4, 5, 6\}$

The elements they have in common are 3 and 5

$A \cap B = \{3, 5\}$

**Example 2**

$A = \{\text{The English alphabet}\}$ $B = \{\text{vowels}\}$

So $A \cap B = \{\text{vowels}\}$
Example 3
A = \{1,2,3,4,5\} \quad B = \{6,7,8,9,10\}
In this case A and B have nothing in common. \( A \cap B = \emptyset \)

3) THE DIFFERENCE:
The difference of two sets A \( \setminus \) B or A - B is those elements which belong to A but which do not belong to B.
A \( \setminus \) B = \{x : x \in A, x \notin B\}

4) COMPLEMENT OF SET:
Complement of set A, denoted by \( A' \) or \( A^c \), is the set of elements which belong to U but which do not belong to A.
A\(^c\) = \{x : x \in U, x \notin A\}

Example: let
A = \{1,2,3\} \quad B = \{3,4\} \quad U = \{1,2,3,4,5,6\}
Find:
A \cup B = \{1,2,3,4\} \quad A \cap B = \{3\} \quad A - B = \{1,2\} \quad A^c = \{4,5,6\}

5) SYMMETRIC DIFFERENCE OF SETS:
The symmetric difference of sets A and B, denoted by A \( \Delta \) B, consists of those elements which belong to A or B but not to both. That is,
A \( \Delta \) B = (A \cup B) \setminus (A \cap B) \quad A \( \Delta \) B = (A \setminus B) \cup (B \setminus A)

Example: Suppose U = N = \{1, 2, 3, \ldots\} is the universal set.
Let A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6, 7\}, C = \{2, 3, 8, 9\}, E = \{2, 4, 6, 8, \ldots\}
Then:
A\(^c\) = \{5, 6, 7, \ldots\}, B\(^c\) = \{1, 2, 8, 9, 10, \ldots\}, C\(^c\) = \{1,4,5,6,7,10,\ldots\} \quad E\(^c\) = \{1, 3, 5, 7, \ldots\}
A \setminus B = \{1,2\}, A \cap C = \{1,4\}, B \cap C = \{4,5,6,7\}, A \cap E = \{1,3\},
B \setminus A = \{5,6,7\}, C \setminus A = \{8,9\}, C \setminus B = \{2,8,9\}, E \setminus A = \{6,8,10,12,\ldots\}.
Furthermore:
A \( \Delta \) B = (A \setminus B) \cup (B \setminus A) = \{1,2,5,6,7\}, \quad B \( \Delta \) C = \{2,4,5,6,7,8,9\},
A \( \Delta \) C = (A \setminus C) \cup (C \setminus A) = \{1,4,8,9\}, \quad A \( \Delta \) E = \{1,3,6,8,10,\ldots\}.
Theorem 2:
A ⊂ B, A ∩ B = A, A ∪ B = B are equivalent

Theorem 3: (Algebra of sets)
Sets under the above operations satisfy various laws or identities which are listed below:

1. \( A ∪ A = A \)
   \( \cap A = A \)

2. \((A ∪ B) ∪ C = A ∪ (B ∪ C)\)
   \((A ∩ B) ∩ C = A ∩ (B ∩ C)\)  
   Associative laws

3. \( A ∪ B = B ∪ A \)
   \( A ∩ B = B ∩ A \)  
   Commutativity

4. \( A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) \)
   \( A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) \)  
   Distributive laws

5. \( A ∪ ∅ = A \)
   \( A ∩ A = A ∩ ∅ \)  
   Identity laws

6. \( A ∪ U = U \)
   \( A ∩ U = A \)  
   Identity laws

7. \((A^c)^c = A \)
   Double complements

8. \( A ∪ A^c = U \)
   \( A ∩ A^c = ∅ \)  
   Complement intersections and unions

9. \( U^c = ∅ \)
   \( ∅^c = U \)

10. \((A ∪ B)^c = A^c ∩ B^c \)
    \((A ∩ B)^c = A^c ∪ B^c \)  
    De Morgan's laws

We discuss two methods of proving equations involving set operations. The first is to break down what it means for an object x to be an element of each side, and the second is to use Venn diagrams.

For example, consider the first of De Morgan's laws:
\( (A ∪ B)^c = A^c ∩ B^c \)

We must prove:
1. \( (A ∪ B)^c \subset A^c ∩ B^c \)
2. \( A^c ∩ B^c \subset (A ∪ B)^c \)

We first show that \( (A ∪ B)^c \subset A^c ∩ B^c \)
Let's pick an element at random $x \in (A \cup B)^c$. We don't know anything about $x$, it could be a number, a function. All we do know about $x$, is that:

$$x \in (A \cup B)^c,$$

so

$$x \notin A \cup B$$

because that's what complement means. Therefore

$$x \notin A \text{ and } x \notin B,$$

by pulling apart the union. Applying complements again we get

$$x \in A^c \text{ and } x \in B^c$$

Finally, if something is in 2 sets, it must be in their intersection, so

$$x \in A^c \cap B^c$$

So, any element we pick at random from $(A \cup B)^c$ is definitely in $A^c \cap B^c$, so by definition

$$(A \cup B)^c \subset A^c \cap B^c$$

Next we show that $(A^c \cap B^c) \subset (A \cup B)^c$.

This follows in a very similar way. Firstly, we pick an element at random from the first set, $x \in (A^c \cap B^c)$

Using what we know about intersections, that means

$$x \in A^c \text{ and } x \in B^c$$

Now, using what we know about complements,

$$x \notin A \text{ and } x \notin B.$$ 

If something is in neither $A$ nor $B$, it can't be in their union, so

$$x \notin A \cup B,$$

And finally

$$\therefore \quad x \in (A \cup B)^c$$

We have proved that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$, and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that these sets have the same elements, i.e. that $(A \cup B)^c = A^c \cap B^c$. 

9
Powerset

The powerset of some set $S$, denoted $P(S)$, is the set of all subsets of $S$ (including $S$ itself and the empty set).

Example 1: Let $A = \{1, 2, 3\}$

Powerset of set $A = P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{\}, A\}$

Example 2: $P(\{0, 1\}) = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}$

Classes of Sets: Collection of subsets of set with some properties Example:

Suppose $A = \{1, 2, 3\}$, let $X$ be the class of subsets of $A$ which contain exactly two elements of $A$. Then

class $X = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

Cardinality

The cardinality of a set $S$, denoted $|S|$, is simply the number of elements a set has. So $|\{a, b, c, d\}| = 4$, and so on. The cardinality of a set need not be finite: some sets have infinite cardinality.

The cardinality of the powerset

Theorem: If $|A| = n$ then $|P(A)| = 2^n$ (Every set with $n$ elements has $2^n$ subsets)

Problemset

Write the answers to the following questions.

1. $|\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}|$
2. $|P(\{1, 2, 3\})|$
3. $P(\{0, 1, 2\})$
4. $P(\{1\})$

Answers

1. 10
2. $2^3 = 8$
3. $\{\{\}, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$
4. $\{\{\}, \{1\}\}$
The Cartesian Product

The Cartesian Product of two sets is the set of all tuples made from elements of two sets. We write the Cartesian Product of two sets \( A \) and \( B \) as \( A \times B \). It is defined as:

\[
A \times B = \{ (a, b) | a \in A \text{ and } b \in B \}
\]

It maybe clearer to understand from examples;

\[
\begin{align*}
\{0, 1\} \times \{2, 3\} &= \{(0, 2), (0, 3), (1, 2), (1, 3)\} \\
\{a, b\} \times \{c, d\} &= \{(a, c), (a, d), (b, c), (b, d)\} \\
\{0, 1, 2\} \times \{4, 6\} &= \{(0, 4), (0, 6), (1, 4), (1, 6), (2, 4), (2, 6)\}
\end{align*}
\]

Example: If \( A = \{1, 2, 3\} \) and \( B = \{x, y\} \) then

A. \( B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\} \)
B. \( A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\} \)

It is clear that, the cardinality of the Cartesian product of two sets \( A \) and \( B \) is:

\[
|A \times B| = |A||B|
\]

A Cartesian Product of two sets \( A \) and \( B \) can be produced by making tuples of each element of \( A \) with each element of \( B \); this can be visualized as a grid (which Cartesian implies) or a table: if, e.g., \( A = \{0, 1\} \) and \( B = \{2, 3\} \), the grid is

| \( \times \) | \( A \) | \( \) | \( B \) |
|---|---|---|
| \( \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) |
| \( 0 \) | \( (0, 2) \) | \( (1, 2) \) | \( (0, 3) \) | \( (1, 3) \) |

Problem set

Answer the following questions:

1. \( \{2,3,4\} \times \{1,3,4\} \)
2. \( \{0,1\} \times \{0,1\} \)
3. \( |\{1,2,3\} \times \{0\}|| \)
4. \( |\{1,1\} \times \{2,3,4\}| \)

Answers

1. \( \{(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)\} \)
2. \( \{(0,0),(0,1),(1,0),(1,1)\} \)
3. 3
4. 6
**Partitionsofset:**
Let S be any nonempty set. A partition \( \prod \) of S is a subdivision of S into nonoverlapping, nonempty subsets. A partition of S is a collection \( \{A_i\} \) of non-empty subsets of S such that:

1) \( A_i \neq \emptyset \), where \( i = 1, 2, 3, \ldots \)
2) The sets of \( \{A_i\} \) are mutually disjoint
   or \( A_i \cap A_j = \emptyset \) where \( i \neq j \).
3) \( \bigcup A_i = S \), where \( A_1 \cup A_2 \cup \ldots \ldots \cup A_n = S \)

The partition of a set into five cells, \( A_1, A_2, A_3, A_4, A_5 \), can be represented by Venn diagram.

Example 1: let \( A = \{1, 2, 3, n\} \)
\( A_1 = \{1\}, A_2 = \{3, n\}, A_3 = \{2\} \)
\( \prod = \{A_1, A_2, A_3\} \) is a partition on \( A \) because it satisfies the three above conditions.

Example 2: Consider the following collections of subsets of \( S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \)
(i) \( \{\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\} \)
(ii) \( \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\} \)
(iii) \( \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\} \)

Then
(i) is not a partition of \( S \) since 7 in \( S \) does not belong to any of the subsets.
(ii) is not a partition of \( S \) since \( \{1, 3, 5\} \) and \( \{5, 7, 9\} \) are not disjoint.
(iii) is a partition of \( S \).

**FINITE SETS, COUNTING PRINCIPLE:**
A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the empty set \( \emptyset \) and the set of letters of English alphabet are finite sets, whereas the set of even positive integers, \( \{2, 4, 6, \ldots\} \), is infinite.
If a set \( A \) is finite, we let \( n(A) \) or \#(A) denote the number of elements of \( A \).
Example: If \( A = \{1, 2, a, w\} \) then
\( n(A) = \#(A) = |A| = 4 \)

Lemma: If \( A \) and \( B \) are finite sets and disjoint then \( A \cup B \) is finite and:
\( n(A \cup B) = n(A) + n(B) \)

**Theorem (Inclusion–Exclusion Principle):** Suppose \( A \) and \( B \) are finite sets. Then
\( A \cup B \) and \( A \cap B \) are finite and
That is, we find the number of elements in \( A \) or \( B \) (or both) by first adding \( n(A) \) and \( n(B) \) (inclusion) and then subtracting \( n(A \cap B) \) (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

**Corollary:**
If \( A, B, C \) are finite sets then
\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]

**Example (1):**
\( A = \{1, 2, 3\} \)
\( B = \{3, 4\} \)
\( C = \{5, 6\} \)

\( A \cup B \cup C = \{1, 2, 3, 4, 5, 6\} \)
\( |A \cup B \cup C| = 6 \)
\( |A| = 3, \ |B| = 2, \ |C| = 2 \)
\( A \cap B = \{3\}, \ |A \cap B| = 1 \)
\( A \cap C = \emptyset, \ |A \cap C| = 0 \)
\( B \cap C = \emptyset, \ |B \cap C| = 0 \)
\( A \cap B \cap C = \emptyset, \ |A \cap B \cap C| = 0 \)

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 3 + 2 + 2 - 1 - 0 - 0 - 0 = 6
\]

**Example (2):**
Suppose a list \( A \) contains the 30 students in a mathematics class, and a list \( B \) contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:
(a) only on list \( A \)
(b) only on list \( B \)
(c) on list \( A \cup B \)

**Solution:**
(a) List \( A \) has 30 names and 20 are on list \( B \); hence \( 30 - 20 = 10 \) names are only on list \( A \).
(b) Similarly, \( 35 - 20 = 15 \) are only on list \( B \).
(c) We seek \( n(A \cup B) \). By inclusion–exclusion,
\[
n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.
\]

**Example (3):**
Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian and:
65 study French (F).
45 study German (G).
42 study Russian (R).
20 students study French & German $F \cap G$.
25 students study French & Russian $F \cap R$.
15 students study German & Russian $G \cap R$.

Find the number of students who study:
1) All three languages ($F \cap G \cap R$)
2) The number of students in each of the eight regions of the Venn diagram

Solution:

$$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R| = 100 = 65$$
$$+45 +42 - 20 - 25 - 15 + |F \cap G \cap R|$$
$$100 = 92 + |F \cap G \cap R|$$

$$\therefore |F \cap G \cap R| = 8 \text{ students study the 3 languages}$$

$$20 - 8 = 12 \quad (F \cap G) - R$$
$$25 - 8 = 17 \quad (F \cap R) - G$$
$$15 - 8 = 7 \quad (G \cap R) - F$$

$$65 - 12 - 8 - 17 = 28 \quad \text{students study French only}$$
$$45 - 12 - 8 - 7 = 18 \quad \text{students study German only}$$
$$42 - 17 - 8 - 7 = 10 \quad \text{students study Russian only}$$
$$120 - 100 = 20 \quad \text{students do not study any language}$$
Mathematical induction:
It is useful for proving propositions that must be true for all integers or for a range of integers.
Proposition: is any statement P(n) which can be either true or false for each n in N. Suppose P has the following two properties.
(i) P(1) is true
(ii) P(k+1) is true whenever P(k) is true
Then P is true for every positive integer \( \forall n \geq k \).

Example 1: Let P be the proposition that the sum of the first n odd numbers is \( n^2 \); that is,

\[ P(n): 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \]

Prove P (for \( n \geq 1 \))

Solution:
(Thenth odd number is 2n – 1, and thenext odd number is 2n +1.) Observethat P(n) is true for n = 1,

(i) \( n=1; \) P(1): \( 2*1 =1^2 \)
(ii) \( n=k; \) Assuming P(k) is true,
We add \( (2k+1) \) to both sides of P(k), obtaining:

\[ 1 + 3 + 5 + \ldots + (2k - 1) + (2k +1) = k^2 + (2k +1) \]
\[ = (k +1)^2 \]

Which is P(k +1). That is, P(k +1) is true whenever P(k) is true. By the principle of mathematical induction, P is true for all \( n \geq k \).

Example 2: Let P be the proposition that the sum of the first n natural numbers is \( \frac{n(n+1)}{2} \)

\[ P(n): 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \]

Prove P (for \( n \geq 1 \))

Solution:
\( n=1 \) (i) P(1): \( \text{left side } =1 \)
\( \text{Right side } =\frac{1*2}{2} =1 \)
(ii) let P(k) is true; \( n=k \)
\[ 1 + 2 + 3 + 4 + \ldots + k = \frac{k(k+1)}{2} \]
to prove that P(k+1) is true

\[ 1 + 2 + 3 + 4 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2} \]
\[ = \frac{k(k+1)+2(k+1)}{2} \]
\[
\frac{(k+1)(k+2)}{2} = \frac{1}{2} (k+1)(k+2)
\]
So P is true for all \( n \geq k \)

**Example 3:**
Prove the following proposition (for \( n \geq 0 \)):

\[ P(n): 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1 \]

**Solution:**
(i) \( P(0) \):

- Left side = 1
- Right side = \( 2^1 - 1 = 1 \)

(ii) Assuming \( P(k) \) is true: \( n=k \)

\[ P(k): 1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1 \]

We add \( 2^{k+1} \) to both sides of \( P(k) \), obtaining

\[ 1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1 \]

which is \( P(k+1) \). That is, \( P(k+1) \) is true whenever \( P(k) \) is true. By the principle of induction, \( P(n) \) is true for all \( n \).

**Homework:**
Prove by induction:

1. \[ 2 + 4 + 6 + \ldots + 2n = n(n+1) \]
2. \[ 1 + 4 + 7 + \ldots + (3n-2) = 1/2 n (3n-1) \]
Relations

Binary relation:
There are many relations in mathematics: "less than", "is parallel to", "is a subset of", etc. These relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. We define a relation simply in terms of ordered pairs of objects.

Product sets:
Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where \( a \in A \) and \( b \in B \) is called the product, or cartesian product, of A and B.

\[ A \times B = \{ (a, b) : a \in A \text{ and } b \in B \} \]

Example: Let \( A = \{1, 2\} \) and \( B = \{a, b, c\} \) then

\[ A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \} \]

Also, \( A \times A = \{ (1, 1), (1, 2), (2, 1), (2, 2) \} \)

The order in which these sets are considered is important, so \( A \times B \neq B \times A \).

Let \( A \) and \( B \) be sets. A binary relation, \( R \), from \( A \) to \( B \) is a subset of \( A \times B \). If \( (x, y) \in R \), we say that \( x \) is \( R \)-related to \( y \) and denote this by \( xRy \).

If \( (x, y) \notin R \), rewrite \( x \not\approx y \) and say that \( x \) is not \( R \)-related to \( y \).

If \( R \) is a relation from \( A \) to \( A \), i.e. \( R \) is a subset of \( A \times A \), then we say that \( R \) is a relation on \( A \).

The domain of a relation \( R \) is the set of all first elements of the ordered pairs which belong to \( R \), and the range of \( R \) is the set of second elements.

Example 1:
Let \( A = \{1, 2, 3, 4\} \). Define a relation \( R \) on \( A \) by writing \( (x, y) \in R \) if \( x < y \). Then \( R = \{ (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \} \).

Example 2:
Let \( A = \{1, 2, 3\} \) and \( R = \{ (1, 2), (1, 3), (3, 2) \} \). Then \( R \) is a relation on \( A \) since it is a subset of \( A \times A \) with respect to this relation:

1R2, 1R3, 3R2 but 1R(1, 1) \( \notin R \) & 2R(2, 1) \( \notin R \)

The domain of \( R \) is \( \{1, 3\} \) and

The range of \( R \) is \( \{2, 3\} \)

Example 3:
Let \( A = \{1, 2, 3\} \). Define a relation \( R \) on \( A \) by writing \( (x, y) \in R \) such that \( a \geq b \), list the element of \( R \)

\[ aRb \leftrightarrow a \geq b, a, b \in A \]

\[ \therefore R = \{ (1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3) \} \]

Example 4:
A relation on the set \( \mathbb{Z} \) of integers is "m divides n." A common notation for this relation is to write \( m \mid n \) when \( m \) divides \( n \). Thus 6 \( \mid 30 \) but 7 \( \nmid 25 \).

Representation of relations:
1) By language
2) By ordered pairs
3) By arrow form
4) By matrix form
5) By coordinates
6) By graph form

Example:
Let \( A=\{1,2,3\} \), the relation \( R \) on \( A \) such that: \( aRb \iff a>b; \quad a,b \in A \)

1) By language:
\[
R = \{(a,b) : a,b \in A \text{ and } aRb \iff a>b\}
\]

2) By ordered pairs
\[
R = \{(2,1),(3,1),(3,2)\}
\]

3) By arrow form

```
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \node (A) at (0,1) {A};
  \node (B) at (2,1) {A};

  \draw[->] (1) to (2);
  \draw[->] (1) to (3);
  \draw[->] (2) to (3);
\end{tikzpicture}
```

4) By matrix form

\[
\begin{array}{ccc}
  \text{1} & \text{2} & \text{3} \\
  \hline
  1 & 0 & 0 & 0 \\
  2 & 1 & 0 & 0 \\
  3 & 1 & 1 & 0 \\
\end{array}
\]

5) By coordinates

```
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \node (A) at (0,1) {1};
  \node (B) at (1,1) {2};
  \node (C) at (2,1) {3};

  \fill (1,1) circle (2pt);
  \fill (2,1) circle (2pt);
  \fill (3,1) circle (2pt);
\end{tikzpicture}
```

6) By graph form
TYPES OF RELATIONS:

Properties of relations:

Let $R$ be a relation on the set $A$

1) Reflexive: $R$ is reflexive if $\forall a \in A \rightarrow aRa \in R$. $\forall a, b \in A$. Thus $R$ is not reflexive if there exists $a \in A$ such that $(a,a) \notin R$.

2) Symmetric: $aRb \rightarrow bRa \forall a, b \in A$. if whenever $(a, b) \in R$ then $(b, a) \in R$. Thus $R$ is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

3) Transitive: $aRb \land bRc \rightarrow aRc$. That is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus $R$ is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

4) Equivalence relation: it is reflexive & symmetric & transitive. That is, $R$ is an equivalence relation on $S$ if it has the following three properties:
   - a) For every $a \in S$, $aRa$.
   - b) If $aRb$, then $bRa$.
   - c) If $aRb$ and $bRc$, then $aRc$.

5) Irreflexive: $\forall a \in A (a,a) \notin R$

6) AntiSymmetric: if $aRb$ and $bRa$ then $a=b$

the relations $\geq, \leq$ and $\subseteq$ are antisymmetric

Example 5: Consider the relation of $C$ of set inclusion on any collection of sets:

1) $A \subset A$ for any set, so $\subset$ is reflexive
2) $A \subset B$ does not imply $B \subset A$, so $\subset$ is not symmetric
3) If $A \subset B$ and $B \subset C$ then $A \subset C$, so $\subset$ is transitive
4) $\subset$ is reflexive, not symmetric & transitive, so $\subset$ is not an equivalence relation.
5) $A \subset A$, so $\subset$ is not Irreflexive
6) If $A \subset B$ and $B \subset A$ then $A = B$, so $\subset$ is anti-symmetric

Example 6: If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 3)\}$

Is $R$ an equivalence relation?

1) $2$ is in $A$ but $(2, 2) \notin R$, so $R$ is not reflexive
2) $(2, 3) \in R$ but $(3, 2) \notin R$, so $R$ is not symmetric
3) $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$, so $R$ is not transitive. So $R$ is not an equivalence relation.
**Example 7**: What are the properties of the relation =?  
1) a = a for any element a ∈ A, so = is reflexive  
2) If a = b then b = a, so = is symmetric  
3) If a = b and b = c, then a = c, so = is transitive  
4) = is (reflexive + symmetric + transitive), so = is equivalence  
5) a = a, so = is not irreflexive  
6) If a = b and b = a, then a = b, so = is antisymmetric  

**Remark**: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation \( R = \{(1, 3), (3, 1), (2, 3)\} \) is neither symmetric nor antisymmetric. On the other hand, the relation \( R = \{(1, 1), (2, 2)\} \) is both symmetric and antisymmetric.

**-Reflexive Closures**
Let \( R \) be a relation on set \( A \). Then:
\[ R \cup \{(a, a) | a \in A\} \] isthe reflexive closure of \( R \). In other words, \( \text{reflexive}(R) \) is obtained by simply adding to \( R \) those elements \((a, a)\) in the diagonal which do not already belong to \( R \).

**-Symmetric Closures**
\( R \cup R^{-1} \) is the symmetric closure of \( R \). In other words, \( \text{symmetric}(R) \) is obtained by adding to \( R \) all pairs \((b, a)\) whenever \((a, b)\) belongs to \( R \).

**EXAMPLE**: Consider the relation \( R = \{(1, 2), (2, 3), (3, 3)\} \) on the set \( A = \{1, 2, 3\} \). Then
\[ \text{reflexive}(R) = R \cup \{(2, 2), (4, 4)\} \]
and
\[ \text{symmetric}(R) = R \cup \{(4, 2), (3, 4)\} \]

**-Transitive Closure**
\( R^* \) is the transitive closure of \( R \), where:
\[ R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n \] and \( R^2 = R \circ R \) and \( R^n = R^{-1} \circ R \).  

**Theorem**: Suppose \( A \) is a finite set with \( n \) elements and let \( R \) be a relation on set \( A \) with \( n \) elements. Then:
\[ \text{transitive}(R) = R + R^2 + R^3 + \ldots + R^n \]

**EXAMPLE**: Consider the relation \( R = \{(1, 2), (2, 3), (3, 3)\} \) on \( A = \{1, 2, 3\} \). Then:
\[ R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\} \] and
\[ R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\} \] then
\[ \text{transitive}(R) = \{(1, 2), (2, 3), (3, 3), (1, 3)\} \]

**Inverse relations:**
\[ R^{-1} = \{(b, a) : (a, b) \in R\} \]

**Example 1**:
Let \( R \) be the following relation on \( A = \{1, 2, 3\} \)
\[ R = \{(1, 2), (1, 3), (2, 3)\} \]
\[ \therefore R^{-1} = \{(2, 1), (3, 1), (3, 2)\} \]

**The matrix for \( R \):**
\[ MR = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \]

and

\[ MR^{-1} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix} \]

MR\(^{-1}\) is the transpose of matrix R

**Composition of relations:**
Let A, B, C be sets and let:
- \( R : A \rightarrow B \) (\( R \subseteq A \times B \))
- \( S : B \rightarrow C \) (\( S \subseteq B \times C \))

There is a relation from A to C denoted by

\( R \circ S \) (composition of R and S): \( A \rightarrow C \)

\[ R \circ S = \{ (a,c) : \exists b \in B \text{ for which } (a,b) \in R \text{ and } (b,c) \in S \} \]

Example: let \( A = \{1,2,3,4\} \)
- \( B = \{a, b, c, d\} \)
- \( C = \{x, y, z\} \)
- \( R = \{(1,a),(2,d),(3,a),(3,d),(3,b)\} \)
- \( S = \{(b,x),(b,z),(c,y),(d,z)\} \)

Find \( R \circ S \)?

Solution:
1) The first way by arrow form

\[ \begin{align*}
2Rd & \quad \text{and} \quad dSz = 2(R \circ S)z \\
\end{align*} \]

and \( 3(R \circ S)x \) and \( 3(R \circ S)z \)
so \( R^\circ S = \{(3,x),(3,z),(2,z)\} \)

2) The second way by matrix:

\[
\begin{align*}
&M_R = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} & & M_S = \begin{bmatrix} x & y & z \\ a & 0 & 0 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

\[
R^\circ S = M_R \cdot M_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 4 & 0 & 0 & 0 \end{bmatrix}
\]

\( R^\circ S = \{(2,z),(3,x),(3,z)\} \)

Theorem 2.1: Let \( A, B, C \) and \( D \) be sets. Suppose \( R \) is a relation from \( A \) to \( B \), \( S \) is a relation from \( B \) to \( C \), and \( T \) is a relation from \( C \) to \( D \). Then \( (R^\circ S) \circ T = R \circ (S \circ T) \)

**n-ARY RELATIONS**

All the relations discussed above were binary relations. By an \textit{n-ary relation}, we mean a set of ordered \( n \)-tuples. For any set \( S \), a subset of the product set \( S^n \) is called an \( n \)-ary relation on \( S \). In particular, a subset of \( S^3 \) is called a ternary relation on \( S \). **EXAMPLE**

(a) Let \( L \) be a line in the plane. Then “betweenness” is a ternary relation \( R \) on the points of \( L \); that is, \((a, b, c) \in R\), if \( b \) lies between \( a \) and \( c \) on \( L \).

(b) The equation \( x^2 + y^2 + z^2 = 1 \) determines a ternary relation \( T \) on the set \( R \) of real numbers. That is, a triple \((x, y, z)\) belongs to \( T \) if \((x, y, z)\) satisfies the equation, which means \((x, y, z)\) is the coordinates of a point in \( R^3 \) on the sphere \( S \) with radius 1 and center \( \text{at the origin } O=(0, 0, 0) \).

**Homework:**

1) Consider the following relations on the set \( A = \{1, 2, 3\} \):

\[
\begin{align*}
R &= \{(1, 1),(1, 2),(1, 3),(3, 3)\}, \\
S &= \{(1, 1),(1, 2),(2, 1),(2, 2),(3, 3)\}, \\
T &= \{(1, 1),(1, 2),(2, 2),(2, 3)\}
\end{align*}
\]

\( \emptyset \) = empty relation

\( A \times A = \) universal relation
Determine whether or not each of the above relations on $A$ is:
(a) reflexive; (b) symmetric; (c) transitive; (d) antisymmetric.

2) for the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A = \{a, b, c\}$.
Find: (a) reflexive($R$); (b) symmetric($R$); (c) transitive($R$).
**Function:**
Function is an important class of relation.

**Definition:**
Let $A, B$ be two nonempty sets, a function $F: A \to B$ is a rule which associates with each element of $A$ a unique element in $B$.

These sets $A$ is called the **domain** of the function, and the set $B$ is called the **range** of the function.

**Example 1:**
Consider the function $f(x) = x^3$, i.e., $f$ assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

**Example 2:**
Consider the following relation on the set $A = \{1, 2, 3\}$
$F = \{(1, 3), (2, 3), (3, 1)\}$
$F$ is a function

$G = \{1, 2, (3, 1)\}$
$G$ is not a function from $A$ to $A$

$H = \{(1, 3), (2, 1), (1, 2), (3, 1)\}$
$H$ is not a function
One-to-one, onto and invertible functions:

1) One-to-one: a function $F: A \rightarrow B$ is said to be one-to-one if different elements in the domain (A) have distinct images. 
   Or if $F(a) = F(a') \Rightarrow a = a'$

2) Onto: $F: A \rightarrow B$ is said to be an onto function if each element of $B$ is the image of some element of $A$.
   $\forall b \in B \quad \exists a \in A: F(a) = b$

3) Invertible (One-to-one correspondence)
   $F: A \rightarrow B$ is invertible if its inverse relation $f^{-1}$ is a function $F: B \rightarrow A$
   $F: A \rightarrow B$ is invertible if and only if $F$ is both one-to-one and onto
   $F^{-1} : \{(b,a) \mid \forall (a,b) \in F\}$

\[
\begin{align*}
& f_1 \colon A \rightarrow B \\
& \text{one-to-one but not onto}(3 \in B \text{ but it is not the image under } f_1)\
\end{align*}
\]
both one to one & onto
(oronetoonecorrespondencebetweenAandB)

\[ f_3 \]
not one to one & onto
Graph of a function:

By a real polynomial function, we mean a function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the $a_i$ are real numbers. Since $\mathbb{R}$ is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The table points are usually obtained from a table where various values are assigned to $x$ and the corresponding value of $f(x)$ computed.

Example 1: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^3$, find $f(x)$

- $f(3) = 3^3 = 27$
- $f(-2) = (-2)^3 = -8$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-27</td>
</tr>
<tr>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>
Example 2: Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( f(x) = x^2 - 2x - 3 \). Find \( f(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
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<td>0</td>
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<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

**Geometrical Characterization of One-to-One and Onto Functions**

For the functions of the form \( f : \mathbb{R} \rightarrow \mathbb{R} \), the graphs of such functions may be plotted in the Cartesian plane, and functions may be identified with their graphs, so the concepts of being one-to-one and onto have some geometrical meaning:

1. \( f : \mathbb{R} \rightarrow \mathbb{R} \) is said to be one-to-one if there are no two distinct pairs \((a_1, b)\) and \((a_2, b)\) in the graph one-to-one or if each horizontal line intersects the graph of \( f \) in at most one point.

2. \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an onto function if each horizontal line intersects the graph of \( f \) at one or more points (at least once).

- \( f_2(x) = 2^x \)

- \( f_3(x) = x^3 - 2x^2 - 5x + 6 \)
(3) If f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f at exactly one point.

\[ f_4(x) = x^3 \]

\[ f_1(x) = x^2 \]

f(x) NOT (ONE-TO-ONE) & NOT (ONTO)

**Composition of function**: Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \), to find the composition function \( g \circ f: A \rightarrow C \)

\[(g \circ f)(a) = g(f(a)) = g(y) = t \]
\[(g \circ f)(b) = g(f(b)) = g(x) = s \]
\[(g \circ f)(c) = g(f(c)) = g(y) = t \]
SEQUENCES OF SETS

A sequence is a function from the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of positive integers into a set $A$. The notation $a_n$ is used to denote the image of the integer $n$. Thus a sequence is usually denoted by $a_1, a_2, a_3, \ldots$

A finite sequence over a set $A$ is a function from $\{1, 2, \ldots, m\}$ into $A$. Such a finite sequence is called a list.

EXAMPLE

(a) The following are two familiar sequences:

1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, \ldots which may be defined by $a_n = \frac{1}{n}$;

1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, \ldots which may be defined by $b_n = 2^{-n}$

Note that the first sequence begins with $n = 1$ and the second sequence begins with $n = 0$.

(b) The sequence $1, -1, 1, -1, \ldots$ may be defined by $a_n = (-1)^n$, where the sequence begins with $n = 0$.

Summation Symbol, Sums

Here we introduce the summation symbol $\sum$ (the Greek letters sigma). Consider a sequence $a_1, a_2, a_3, \ldots$. Then we define the following:

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

EXAMPLE:

$$\sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

$$\sum_{j=1}^{n} j = 1 + 2 + \cdots + n = n(n+1)/2, \quad \text{for example,} \quad 1 + 2 + \cdots + 50 = (50 \times 51)/2 = 1275$$

RECURSIVELY DEFINED FUNCTIONS

A function is said to be recursively defined if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

1. There must be certain arguments, called base values, for which the function does not refer to itself.
2. Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be well-defined.
Factorial Function

The product of the positive integers from 1 to \( n \), inclusive, is called “\( n \) factorial” and is usually denoted by \( n! \). That is,

\[
n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1
\]

where \( 0! = 1 \), so that the function is defined for all nonnegative integers. Thus:

\[
0! = 1, \\
2! = 2 \cdot 1 = 2, \\
3! = 3 \cdot 2 \cdot 1 = 6, \\
4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720
\]

This is true for every positive integer \( n \); that is,

\[
n! = n \cdot (n-1)!
\]

Accordingly, the factorial function may also be defined as follows:

Definition of Factorial Function:

(a) If \( n = 0 \), then \( n! = 1 \).

(b) If \( n > 0 \), then \( n! = n \cdot (n-1)! \)

The definition of \( n! \) is recursive, since it refers to itself when it uses \( (n-1)! \). However:

(1) The value of \( n! \) is explicitly given when \( n = 0 \) (thus 0 is a base value).

(2) The value of \( n! \) for arbitrary \( n \) is defined in terms of a smaller value of \( n \) which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

EXAMPLE: the 4! Can be calculated in 9 steps using the recursive definition.

(1) \( 4! = 4 \cdot 3! \)

(2) \( 3! = 3 \cdot 2! \)

(3) \( 2! = 2 \cdot 1! \)

(4) \( 1! = 1 \cdot 0! \)

(5) \( 0! = 1 \)

(6) \( 1! = 1 \cdot 1 = 1 \)

(7) \( 2! = 2 \cdot 1 = 2 \)

(8) \( 3! = 3 \cdot 2 = 6 \)

(9) \( 4! = 4 \cdot 6 = 24 \)

Fibonacci Sequence

The Fibonacci sequence (usually denoted by \( F_0, F_1, F_2, \ldots \)) is as follows:

\[
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots
\]

That is, \( F_0 = 0 \) and \( F_1 = 1 \) and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

\[
34 + 55 = 89 \text{ and } 55 + 89 = 144
\]

Fibonacci Sequence can be defined:
(a) If $n = 0$, or $n = 1$, then $F_n = n$.
(b) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

Where: The base values are 0 and 1, and the value of $F_n$ is defined in terms of smaller values of $n$ which are closer to the base values. Accordingly, this function is well-defined.
Logic and propositional calculus

A proposition (or statement) is a declarative statement which is true or false, but not both. Example: the following six sentences:

1. Ice floats in water.
2. China is in Europe.
3. $2 + 2 = 4$
4. $2 + 2 = 5$
5. Where are you going?
6. Do your homework.

The first four are propositions, the last two are not. Also, (1) and (3) are true, but (2) and (4) are false.

Compound Propositions

It is the proposition that composed of subpropositions and various connectives. Primitive proposition is the proposition that cannot be broken down into simpler propositions. For example, the above propositions are primitive propositions, while:

“Roses are red and violets are blue.” and
“John is smart or he studies every night.” are compound.

Basic Logical Operations

1. Conjunction, $p \land q$
2. Disjunction, $p \lor q$
3. Negation, $\neg p$

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
<th>$p$</th>
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(a) “$p$ and $q$”  
(b) “$p$ or $q$”  
(c) “not $p$”

Example: Consider the following four statements:

(i) Ice floats in water and $2 + 2 = 4$.
(ii) Ice floats in water and $2 + 2 = 5$.
(iii) China is in Europe and $2 + 2 = 4$.
(iv) China is in Europe and $2 + 2 = 5$.

Only the first statement is true. Each of the others is false since at least one of its sub-statements is false.

Example: Consider the following four statements:

(i) Ice floats in water or $2 + 2 = 4$.
(ii) Ice floats in water or $2 + 2 = 5$.
(iii) China is in Europe or $2 + 2 = 4$.
(iv) China is in Europe or $2 + 2 = 5$. 
Only the last statement (iv) is false. Each of the others is true since at least one of its sub-statements is true.

**EXAMPLE:** Consider the following six statements:

(a1) Ice floats in water.  
(a2) It is false that ice floats in water.  
(a3) Ice does not float in water.  
(b1) 2 + 2 = 5  
(b2) It is false that 2 + 2 = 5.  
(b3) 2 + 2 ≠ 5

Then (a2) and (a3) are each the negation of (a1); and (b2) and (b3) are each the negation of (b1). Since (a1) is true, (a2) and (a3) are false; and since (b1) is false, (b2) and (b3) are true.

**PROPOSITIONS AND TRUTH TABLES**

The truth table for the compound proposition \( \neg(p \land \neg q) \) is:

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<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( p \land \neg q )</th>
<th>( \neg(p \land \neg q) )</th>
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**TAUTOLOGIES AND CONTRADICTIONS**

Some propositions \( P(p, q, \ldots) \) contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies. Analogously, a proposition \( P(p, q, \ldots) \) is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example, the proposition “p or not p,” that is, \( p \lor \neg p \), is a tautology, and the proposition “p and not p,” that is, \( p \land \neg p \), is a contradiction.

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<th>( p )</th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
<th>( p \land \neg p )</th>
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(a) \( p \lor \neg p \)  
(b) \( p \land \neg p \)

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

**LOGICAL EQUIVALENCE**

Two propositions \( P(p, q, \ldots) \) and \( Q(p, q, \ldots) \) are said to be logically equivalent, or equal, denoted by \( P(p, q, \ldots) \equiv Q(p, q, \ldots) \), if they have identical truth tables.
for example, the truth tables of \( \neg(p \land q) \) and \( \neg p \lor \neg q \), both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write:
\[
\neg(p \land q) \equiv \neg p \lor \neg q
\]

## ALGEBRA OF PROPOSITIONS

### CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, particularly in mathematics, are of the form “If \( p \) then \( q \).” Such statements are called conditional statements and are denoted by: \( p \rightarrow q \)

The conditional \( p \rightarrow q \) is frequently read “\( p \) implies \( q \)” or “\( p \) only if \( q \).”

Another common statement is of the form “\( p \) if and only if \( q \).” Such statements are called biconditional statements and are denoted by \( p \iff q \)

\[
\begin{array}{|c|c|c|c|}
\hline
p & q & p \land q & \neg(p \land q) \\
\hline
T & T & T & F \\
T & F & F & T \\
F & T & F & T \\
F & F & F & T \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
p & q & \neg p & \neg q & \neg p \lor \neg q \\
\hline
T & T & F & F & F \\
T & F & F & T & T \\
F & T & T & F & T \\
F & F & T & T & T \\
\hline
\end{array}
\]

(a) \( \neg(p \land q) \)  
(b) \( \neg p \lor \neg q \)
(a) The conditional $p \rightarrow q$ is false only when the first part $p$ is true and the second part $q$ is false. Accordingly, when $p$ is false, the conditional $p \rightarrow q$ is true regardless of the truth value of $q$.

(b) The biconditional $p \iff q$ is true whenever $p$ and $q$ have the same truth values and false otherwise.

Note that the truth table of $p \rightarrow q$ and $\neg p \lor q$ are identical, that is, they are both false only in the second case. Accordingly, $p \rightarrow q$ is logically equivalent to $\neg p \lor q$; that is, $p \rightarrow q \equiv \neg p \lor q$.

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(c) $\neg p \lor q$

(a) $p \rightarrow q$