SETSANDELEMENTS

Aset is a collection ofobjects called theelementsormembers oftheset. Theordering ofthe elements is not important and repetition of elements is ignored, for example{1, 3, 1, 2, 2, 1} ={1, 2, 3}.
Oneusuallyuses capital letters, A,B,X, Y, . . . , todenotesets, and lowercaseletters, a, b, x,y, . . ., to denoteelements ofsets.
Belowyou'll seejust asamplingofitems that could be considered as sets:

- Theitems in astore
- The English alphabet
- Even numbers

Aset could haveas manyentries asyou would like.
It could haveoneentry, 10 entries, 15 entries, infinitenumberof entries, orevenhave no entries at all!
For example, in the abovelist the English alphabet would have26 entries,whilethe set of even numbers would have an infinitenumberof entries.

Each entryinaset is known as an **elementormember**

Sets are written usingcurlybrackets"{"and"}",with their elements listed in between.
For examplethe English alphabet could bewrittenas
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}

and even numbers couldbe{0,2,4,6,8,10,...} (Note: thedots at the end indicatingthat theset goes on infinitely)

Principles:

- ∈ belongto
- ∉ not belongto
- ⊆ subset
- ⊂ propersubset,For example, {a, b} is apropersubset of{a, b, c}, but {a, b,c} is not apropersubset of{a, b, c}.
- ⊄ not subset

So we could replacethestatement "ais belongto the alphabet"with
a ∈{alphabet} and replacethestatement"3 is notbelongto theset of evennumbers" with3 ∉{Even numbers}

Nowif wenamed oursets we couldgo even further.
Givetheset consistingofthe**alphabet**thename A, andgivetheset consistingof **even numbers** thenameE.

We could now write
a ∈A
and
3∉E.

Problem

Let A={2, 3, 4, 5} andC ={1, 2, 3, . . ., 8, 9},Showthat Ais apropersubset ofC.

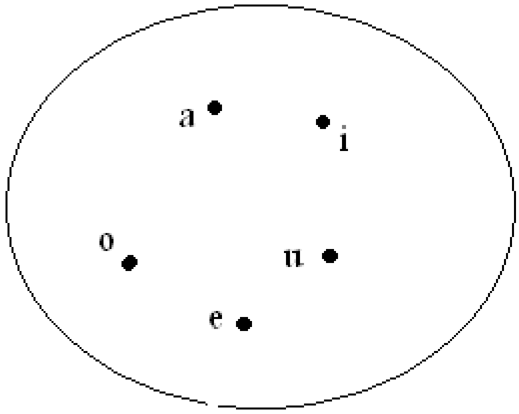
Answer

Each element ofAbelongs toC so A⊆C. On theotherhand, 1∈C but 1∉A.
HenceA≠C. ThereforeAis apropersubset ofC.

Therearethreeways to specifyaparticularset:

1) Bylist its members separated bycommas andcontained in braces{ }, (ifit is possible), for example, A={a,e,i,o,u}
2) Bystatethosepropertieswhich characterizethe elements in theset, for example, A={x:xis aletterin the English alphabet, xis avowel}
3) Venn diagram: (Agraphical representation ofsets).

A



Example (1)

A={x:xis aletterin the English alphabet, xis avowel} e
∈A(eisbelongto A)
f ∉A (fisnot belongtoA)

Example (2)
Xis theset {1,3,5,7,9}
3∈X and 4∉X

Example (3)
Let E={x|x^2−3x+2 =0} → (x-2)(x-1)=0 → x=2 &x=1
E={2, 1}, and 2∈□

Universalset,emptyset:

Inanyapplication ofthetheoryofsets, themembers of all sets underinvestigation usuallybelongto some fixed largeset called theuniversal set. For example, in human population studies theuniversal set consists of allthepeoplein the world.We will let thesymbol Udenotes theuniversal set.
Thesetwithnoelements is calledtheemptysetor nullsetandisdenotedby∅or{}

Subsets:

Everyelement in asetAis also an element ofaset B, then Ais called asubset ofB. We also saythatBcontains A.This relationship is written:

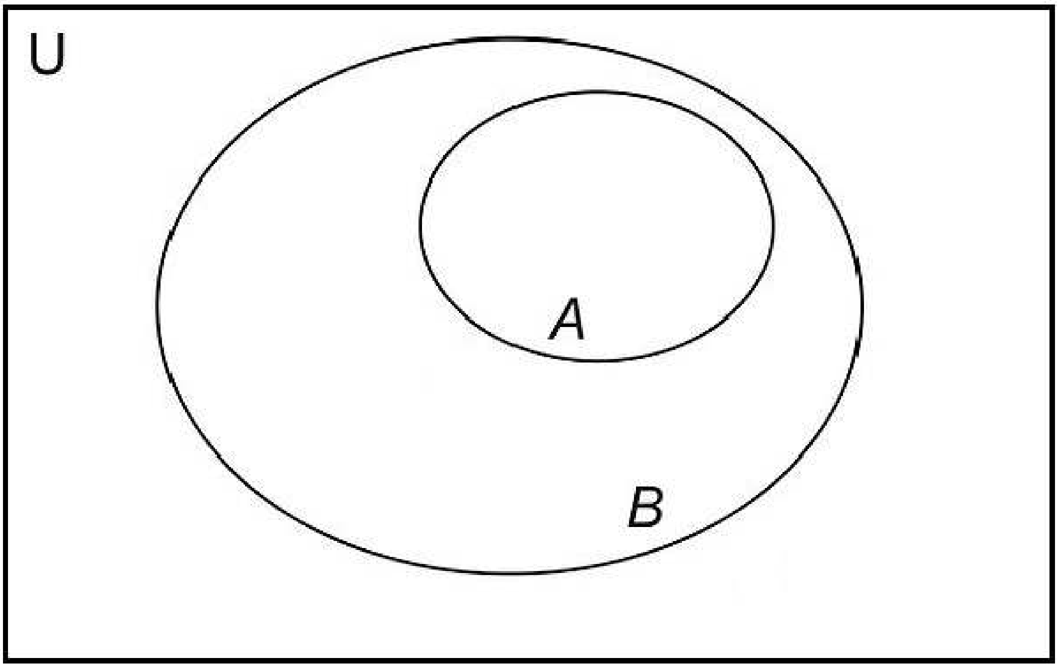
A⊂B or B⊃A

IfAis not asubset ofB,i.e. if at least oneelement ofAdosenot belongto B, we writeA⊄B.
Example4:
Considerthesets.
A={1,3,4,5,8,9} B ={1,2,3,5,7} and C ={1,5}
ThenC⊂A and C⊂Bsince1 and 5, theelementofC, are also members ofA andB. ButB⊄Asincesomeofits elements, e.g. 2and7,do not belongto A.Furthermore, sincethe elements ofA,Band C must also belongto theuniversal set U,wehavethat Umust at least theset {1,2,3,4,5,7,8,9}.

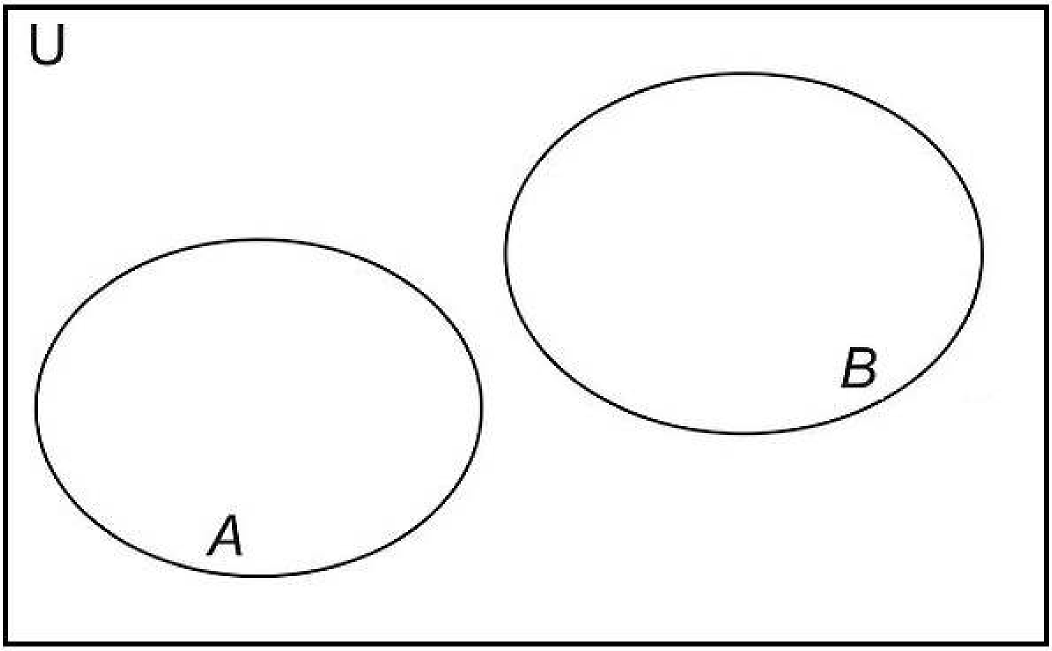
$A \subset B : \{\forall x \in A \quad \Rightarrow \quad x \in B$
$A \not\subset B : \{\exists x \in A \quad \text{but} \quad x \notin B$

$\forall$:Forall لكل
$\exists$:Thereexists يوجد على الاقل

Thenotion ofsubsets is graphicallyillustrated below:



Ais entirelywithin BsoA$\subset$B.



*A*and*B*aredisjointor(A∩B=∅)so wecouldwriteA⊄BandB⊄A.

Setofnumbers:

Several sets areused so often, theyaregiven special symbols.

N= theset of*natural numbers* orpositiveintegers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Z= theset ofall integers: . . . ,−2,−1, 0, 1, 2, . . .

$$\mathbb{Z} = \mathbb{N} \cup \{\ldots, -2, -1\}$$

Q = theset of rational numbers

$$\mathbb{Q} = \mathbb{Z} \cup \{\ldots, -1/3, -1/2, 1/2, 1/3, \ldots, 2/3, 2/5, \ldots\}$$

Where Q={a/b:a,b∈Z,b≠0}

R= theset ofreal numbers

$$\mathbb{R} = \mathbb{Q} \cup \{\ldots, -\pi, -\sqrt{2}, \sqrt{2}, \pi, \ldots\}$$

C= theset of complexnumbers

$$\mathbb{C} = \mathbb{R} \cup \{i, 1+i, 1-i, \sqrt{2}+\pi i, \ldots\}$$

Where C={x+iy;x,y∈R;i=√-1} Observethat **N**

⊂**Z**⊂**Q** ⊂**R**⊂**C**.

Theorem1:

For anyset A, B, C:

1- $\varnothing \subset A \subset U$.

2- $A \subset A$.

3- IfA⊂B andB ⊂C, thenA⊂C.

4- A=Bifand onlyifA ⊂BandB⊂A.

Setoperations:

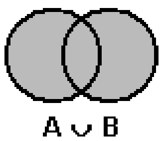
1) UNION:

The*union*of twosets*A*and*B*,denotedby$A \cup B$,isthesetof allelementswhich belongto *A*orto *B*;

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Example

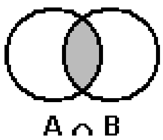
A={1,2,3,4,5} B={5,7,9,11,13}
A∪B={1,2,3,4,5,7,9,11,13}



2)INTERSECTION

The*intersection*oftwosets*A* and*B*,denotedby$A \cap B$,isthesetofelementswhichbelongtoboth*A* and*B*;

$$A \cap B = \{x : x \in A \text{and} x \in B\}.$$



Example1

A={1,3,5,7,9}   B={2,3,4,5,6}
The elements theyhavein common are3 and 5
A∩B={3,5}

Example2
A={The English alphabet}   B={vowels}
SoA∩B={vowels}

Example3

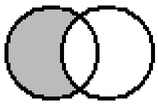
A={1,2,3,4,5}   B={6,7,8,9,10}

InthiscaseA andBhave nothingin common.$A \cap B = \varnothing$

3) THE DIFFERENCE:

Thedifferenceoftwo sets A\Bor A-Bis those elements which belongtoAbut which do not belongtoB.

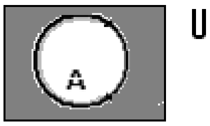
$A \setminus B = \{x : x \in A,\ x \notin B\}$



A – B

4) COMPLEMENT OFSET: Complement ofset A^cor A', is theset ofelements which belongto Ubut which do not belongto A.

$A^c = \{x : x \in U,\ x \notin A\}$



Example: let   A={1,2,3}

B={3,4}

U={1,2,3,4,5,6}

Find:

$A \cup B = \{1,2,3,4\}$

$A \cap B = \{3\}$

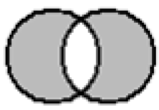
A-B ={1, 2}

$A^c = \{4, 5, 6\}$

5) Symmetricdifferenceofsets

Thesymmetricdifferenceofsets A andB, denoted by $A \oplus B$, consists ofthose elements which belongto AorBbut not to both. That is,

$A \oplus B = (A \cup B) \setminus (A \cap B)$ or $A \oplus B = (A \setminus B) \cup (B \setminus A)$



A ⊕ B

Example: SupposeU=N={1, 2,3, .. .}istheuniversalset.

Let A={1, 2, 3, 4}, B={3, 4, 5, 6, 7},C={2, 3, 8, 9},E={2, 4, 6, 8,. . .}

Then:

$A^c = \{5, 6, 7, \ldots\}$, $B^c = \{1, 2, 8, 9, 10, \ldots\}$, $C^c = \{1,4,5,6,7,10,\ldots\}$ $E^c = \{1, 3, 5, 7, \ldots\}$

A\B={1, 2}, A\C ={1,4}, B\C={4, 5, 6, 7}, A\E={1, 3},

B\A={5, 6, 7}, C\A={8, 9}, C\B ={2, 8, 9}, E\A={6, 8, 10, 12, . . .}.

Furthermore:

$A \oplus B = (A \setminus B) \cup (B \setminus A) = \{1,2,5,6,7\}$, $B \oplus C = \{2, 4, 5, 6, 7, 8, 9\}$,

$A \oplus C = (A \setminus C) \cup (B \setminus C) = \{1,4,8,9\}$, $A \oplus E = \{1, 3, 6, 8, 10, \ldots\}$.

Theorem2 :
$A \subset B, A\cap B=A, A\cup B=B$ are equivalent

Theorem3:(Algebra ofsets)
Sets underthe aboveoperations satisfyvarious laws oridentities which arelisted below:

1- $A\cup A=AA$
$\cap A=A$

2- $(A\cup B)\cup C=A\cup(B\cup C)$ Associativelaws
$(A\cap B)\cap C=A\cap (B\cap C)$

3- $A\cup B=B\cup A$ Commutativity
$A\cap B=B\cap A$

4- $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ Distributivelaws
$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

5- $A\cup\varnothing=A$ Identitylaws
$A\cap U=A$

6- $A\cup U=U$ Identitylaws
$A\cap\varnothing=\varnothing$

7- $(A^c)^c=A$ Double complements

8- $A\cup A^c=U$ Complement intersections and unions

$A\cap A^c=\varnothing$

9- $U^c=\varnothing$
$\varnothing^c=U$

10- $(A\cup B)^c=A^c\cap B^c$ DeMorgan's laws
$(A\cap B)^c=A^c\cup B^c$

Wediscuss two methods ofproving equations involvingset operations. The first is to break downwhat it means for an object xto be anelement ofeach side, andthe second is to use Venn diagrams.
For example, considerthe first of DeMorgan's laws:

$$(A\cup B)^c=A^c\cap B^c$$

Wemust prove: 1) $(A\cup B)^c\subset A^c\cap B^c$
2) $A^c\cap B^c\subset(A\cup B)^c$

We first showthat $(A\cup B)^c\subset A^c\cap B^c$

Let's pickanelementat random $x \in (A \cup B)^c$. We don'tknowanything aboutx,it could beanumber,a function. All wedo know about x, isthat:

$x \in (A \cup B)^c$,so
$x \notin A \cup B$

becausethat'swhat complement means. Therefore

$x \notin A$and $x \notin B$,

bypulling apart theunion. Applyingcomplements again weget

$x \in A^c$ and $x \in B^c$

Finally, ifsomethingis in 2 sets, it must bein theirintersection, so

$x \in A^c \cap B^c$

So,anyelementwe pickatrandomfrom:$(A \cup B)^c$ isdefinitelyin,$A^c \cap B^c$, so bydefinition

$(A \cup B)^c \subset A^c \cap B^c$

Next weshowthat $(A^c \cap B^c) \subset (A \cup B)^c$.

This follows averysimilar way.Firstly,wepickan element at random from the first set,$x \in (A^c \cap B^c)$

Usingwhat weknowabout intersections, that means

$x \in A^c$ and$x \in B^c$

Now, usingwhat weknow about complements,

$x \notin A$ and $x \notin B$.

Ifsomethingis in neitherAnorB, it can't bein theirunion, so

$x \notin A \cup B$,

And finally

$\therefore \quad x \in (A \cup B)^c$

Wehaveprovethateveryelement of $(A \cup B)^c$ belongsto $A^c \cap B^c$andthat everyelementof $A^c \cap B^c$ belongsto $A \cup B)^c$ .Together,these inclusionsprove that thesets havethesame elements, i.e. that $(A \cup B)^c = A^c \cap B^c$

Powerset

Thepowerset ofsomeset S, denoted P(S), is theset of all subsets ofS (includingS itself and the emptyset)

Example1: Let A={ 1,23}

Powerset ofset A=P(A)=[{1},{2},{3},{1,2},{1,3},{2,3},{},A]

Example2: P({0,1})={{},{0},{1},{0,1}}

Classes ofsets:

Collection ofsubset ofaset with someproperties Example:

SupposeA={ 1,2 3} , let Xbethe class ofsubsets ofA which contain exactlytwo elements ofA. Then

class X=[{1,2},{1,3},{2,3}]

Cardinality

The cardinalityofaset S, denoted**|S|**, is simplythenumberofelements aset has. So **|**{a,b,c,d}**|=** 4,and so on.The cardinalityofaset need not be finite: somesets have infinite cardinality.

Thecardinality ofthe powerset

Theorem:If|A|=n then|P(A)|=2^n(Everyset with n elements has 2^nsubsets)

Problemset

writethe answers to thefollowingquestions.

1. |{1,2,3,4,5,6,7,8,9,0}|
2. |P({1,2,3})|
3. P({0,1,2})
4. P({1})

Answers

1. 10
2. 2^3=8
3. {{},{0},{1},{2},{0,1},{0,1,2},{0,2},{1,2}}
4. {{},{1}}

TheCartesianproduct

TheCartesian Product oftwo sets is theset of all tuples made from elements oftwo sets. We writetheCartesian Product oftwo sets Aand BasA×B.It is defined as:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

It maybe clearerto understand from examples;

$$\{0,1\} \times \{2,3\} = \{(0,2),(0,3),(1,2),(1,3)\}$$
$$\{a,b\} \times \{c,d\} = \{(a,c),(a,d),(b,c),(b,d)\}$$
$$\{0,1,2\} \times \{4,6\} = \{(0,4),(0,6),(1,4),(1,6),(2,4),(2,6)\}$$

Example:IfA={1, 2, 3} and B ={x,y} then
A. B={(1, x), (1,y), (2, x), (2,y), (3, x), (3,y)}
B. A={(x, 1), (x, 2), (x,3), (y, 1), (y, 2), (y, 3)}

It is clearthat, thecardinalityoftheCartesian product oftwo sets A andB is:

$$|A \times B| = |A||B|$$

ACartesian Product oftwo sets A andBcan beproduced bymakingtuples of each element ofA with each element ofB; this can bevisualized as agrid(which *Cartesian* implies)ortable: if, *e.g.*, A={ 0, 1 } and B={ 2, 3 }, thegrid is

<table>
<tr><th colspan="2" rowspan="2">×</th><th colspan="2">A</th></tr>
<tr><th>0</th><th>1</th></tr>
<tr><th rowspan="2">B</th><th>2</th><td>(0,2)</td><td>(1,2)</td></tr>
<tr><th>3</th><td>(0,3)</td><td>(1,3)</td></tr>
</table>

Problemset

Answerthefollowingquestions:

1. {2,3,4}×{1,3,4}

2.{0,1}×{0,1}

3. |{1,2,3}×{0}|
4. |{1,1}×{2,3,4}|

Answers

1. {(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)}
2. {(0,0),(0,1),(1,0),(1,1)}
3. 3
4. 6

Partitionsofset:

Let S beanynonemptyset. A partition (∏)ofSis asubdivisionofS into nonoverlapping, nonemptysubsets. Apartition ofS is a collection {Ai} ofnon-empty subsets ofS such that:

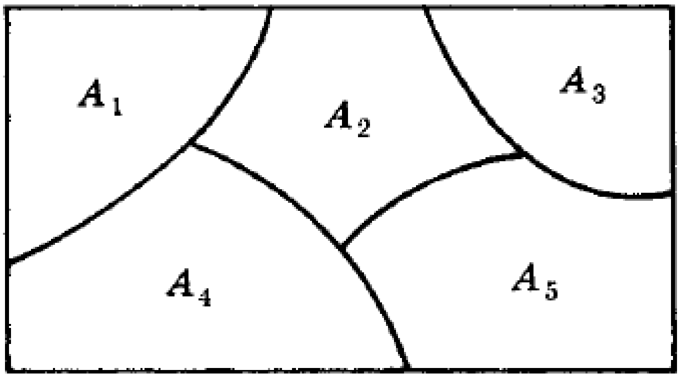
1)$A_i \neq \varnothing$, wherei=1,2,3,……

2) Thesets of{Ai }aremutuallydisjoint

or $A_i \cap A_j = \varnothing$ wherei≠j.

3)$\cup A_i = S$, where $A_1 \cup A_2 \cup \ldots \cup A_i = S$

Thepartition ofaset intofive cells, A1, A2,A3,A4,A5,can berepresented byVenn diagram



Example1: let A={1,2,3,n}

A1 ={1},A2={3,n}, A3 ={2}

∏={A1, A2, A3} isapartition on Abecauseit satisfythethreeabove conditions.

Example2 : Considerthe following collectionsofsubsets ofS ={1,2,3,4,5,6,7,8,9} (i) [{1,3,5},{2,6},{4,8,9}]

(ii) [{1,3,5},{2,4,6,8},{5,7,9}]

(iii) [{1,3,5},{2,4,6,8},{7,9}]

Then

(i) is not apartition ofS since7 in S does not belongto anyofthesubsets.

(ii) is not apartition ofS since{1,3,5} and {5,7,9} arenot disjoint.

(iii) is apartition ofS.

FINITE SETS, COUNTING PRINCPLE:

Aset is said to be finiteifit contains exactlym distinct elements wheremdenotes somenonnegativeinteger. Otherwise,aset is saidto beinfinite. For example, the emptyset∅andthesetoflettersofEnglishalphabetare finitesets,whereasthesetof even positiveintegers, {2,4,6,…..}, is infinite.

Ifaset Ais finite, welet *n*(A)or#(A)denotethenumberofelements ofA.

Example:IfA ={1,2,a,w} then

n(A)=#(A)=|A|=4

Lemma:IfAandBarefinitesetsanddisjointThenA∪Bisfinitesetand:

$$n(A \cup B) = n(A) + n(B)$$

Theorem(Inclusion–Exclusion Principle): SupposeA andBare finitesets. Then A ∪B andA∩Barefiniteand

$$|A \cup B| = |A| + |B| - |A \cap B|$$

That is, we find thenumberof elements in AorB (orboth)byfirst adding n(A) and n(B)(inclusion) and thensubtractingn($A \cap B$)(exclusion)sinceits elements were counted twice.
We can applythis result to obtain asimilar formula forthreesets:

Corollary:
IfA,B,C arefinitesets then
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example (1):
A={1,2,3}
B={3,4}
C={5,6}

$A \cup B \cup C = \{1,2,3,4,5,6\}$
$|A \cup B \cup C| = 6$

$|A| = 3$, $|B| = 2$, $|C| = 2$
$A \cap B = \{3\}$, $|A \cap B| = 1$
$A \cap C = \{\}$, $|A \cap C| = 0$
$B \cap C = \{\}$, $|B \cap C| = 0$
$A \cap B \cap C = \{\}$, $|A \cap B \cap C| = 0$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 3+2+2-1-0-0+0 = 6$

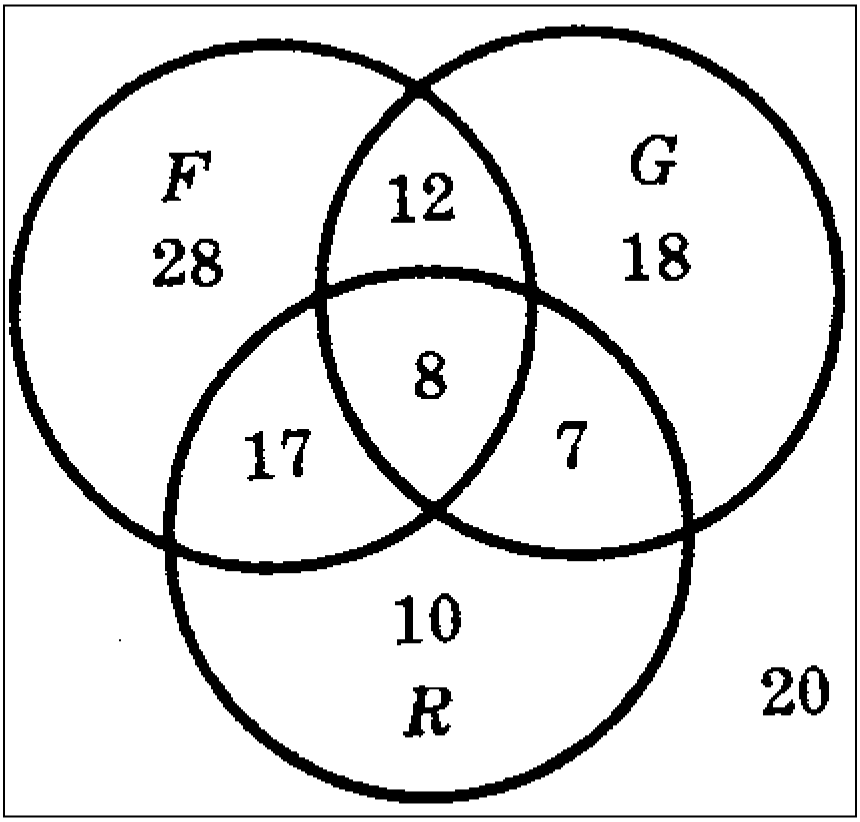
Example (2):
Supposealist A contains the30 students in amathematics class, andalist Bcontains the35 students in an English class, and supposethereare20 names on both lists. Find thenumberofstudents:
(a)onlyon list A
(b)onlyon list B
(c)onlist$A \cup B$

Solution:
(a) List Ahas 30 names and 20 areon list B; hence30 −20 =10 namesareonlyon list A.
(b) Similarly, 35 −20=15 areonlyon list B.
(c) Weseekn($A \cup B$).By inclusion–exclusion,
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

Example (3):
Supposethat 100 of120 computersciencestudents at a collegetake at least oneof languages: French,German, and Russian and:
65 studyFrench(F).
45 studyGerman (G).
42 studyRussian (R).

20studyFrench&GermanF∩G.
25studyFrench& RussianF∩R.
15studyGerman& RussianG∩R.
Find thenumberofstudents who study:

1) Allthreelanguages(F∩G∩R)
2) Thenumberofstudents in each ofthe eight regions ofthe Venn diagram



Solution:
|F ∪G ∪R|=|F|+|G|+|R| -|F∩G| -|F∩R|-|G∩R| +|F∩G∩R| 100 =65
+45 +42 - 20 - 25 - 15 +|F∩G∩R|
100 =92 +|F∩ G∩R|
∴|F∩ G∩ R|=8studentsstudythe3languages

20 – 8 =12 (F∩G)-R
25 – 8 =17 (F∩R)-G
15 – 8 =7 (G∩R)-F

65 – 12 – 8 – 17 =28 students studyFrench only
45 – 12 – 8 7 =18 students studyGerman only
42 – 17 – 8 7 =10 students studyRussian only
120 – 100 =20 students do not studyanylanguage

Mathematicinduction:

It is usefulforprovingpropositions that must betrue for all integers or forarangeof integer.

Proposition: is anystatement P(n) which can beeithertrueorfalse foreach n in N.

SupposeP has the followingtwo properties.

(i) P(1)is true

(ii) P(k+1)is true wheneverP(k)is true

ThenPistrueforeverypositiveinteger $\forall n \geq k$.

Example1: Let P betheproposition that thesumofthe first n odd numbers is n^2; that is,

$$P(n): \quad 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

ProveP (forn ≥ 1)

Solution:

(Thenth odd numberis 2n – 1, and thenext odd numberis 2n +1.) Observethat P(n) is true forn =1,

(i) n=1; P(1): $2*n-1 = 1^2$

(ii) n=k; AssumingP(k)is true,

We add (2k-1)+2 =2K+1 to bothsides ofP(k), obtaining:

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$
$$= (k + 1)^2$$

Which is P(k +1). That is, P(k +1)is true wheneverP(k)is true. Bytheprincipleof mathematicalinduction,Pistrueforalln≥.k.

Example2:.

$$P(n): \quad 1 + 2 + 3 + 4 + \ldots + n = 1/2\, n(n+1)$$

or $$\sum_{i=1}^{n} i = 1/2\, n\,(n + 1)$$

ProveP (forn ≥ 1)

solution :

n =1 (i) P(1): left side =1

Right side =1/2 * 1 * (2) =1

(ii)let P(k)is true; n=k

$$1 + 2 + 3 + 4 + \ldots + k = 1/2 * k * (k+1)$$

to provethat P(k+1)is true

$$1 + 2 + 3 + 4 + \ldots + k + (k+1) = 1/2 * k * (k+1) + (k+1)$$

$$= \frac{k\,(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$=1/2\,(k+1)(k+2)$$

SoPistrueforalln≥k

Example3:

Provethe followingproposition (forn ≥ 0):

$$P(n):\ 1+2+2^2+2^3+\ldots+2^n=2^{n+1}-1$$

solution :

(i) P(0): leftside =1

Right side $=2^1-1=1$

(ii) AssumingP(k)is true; n=k

$$P(k):\ 1+2+2^2+2^3+\ldots+2^k=2^{k+1}-1$$

We add 2k+1 to both sides ofP(k), obtaining

$$1+2+2^2+2^3+\ldots+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$$
$$=2(2^{k+1})-1=2^{k+2}-1$$

which is P(k +1). That is,P(k +1)is true wheneverP(k)is true. Bytheprincipleof induction, P(n)is true forall n.

Homework:

Provebyinduction:

1) 2 +4 +6 +…….+2n =n (n +1)
2) 1 +4 +7 +…….+ (3n –2)=1/2 n (3n-1)

Relations

Binary relation:

Therearemanyrelationsin mathematics :"less than","is parallel to","is asubset of", etc. Theserelations considerthe existenceornonexistenceofacertain connection between pairs ofobjects taken in adefiniteorder.Wedefinearelation simplyin terms ofordered pairs ofobjects.

Productsets:

Considertwo arbitrarysets A and B. Theset of allordered pairs (a,b) wherea∈Aand b∈Bis called theproduct, or cartesianproduct, ofA and B.

$A\times B = \{(a,b): a\in A \text{ and } b\in B\}$

Example:Let A={1,2}and B ={a,b ,c} then

$A\times B = \{(1,a), (1,b),(1,c),(2,a),(2,b),(2,c)\}$

Also, $A\times A=\{(1, 1), (1, 2),(2, 1), (2, 2)\}$

- Theorderin which thesets are considered is important,so $A\times B \neq B \times A$.

LetAandBbesets. Abinaryrelation, R, from Ato Bis asubset ofA×B.If(x,y)∈R, wesaythat xis R-relatedtoyand denotethis by xRy

if(x,y)∉R, wewrite $x \not R y$ and saythat xis not R-related toy.

ifR is a relation from Ato A,i.e. R is asubset ofA× A, then wesaythatR is a relation on A.

The**domain** ofa relationR is theset of all first elements oftheordered pairs which belongto R, and the**range**ofR is theset ofsecond elements.

Example1:

LetA={1, 2, 3, 4}. Definea relation R on Abywriting(x,y)∈R ifx<y.Then R ={(1, 2), (1, 3),(1, 4), (2, 3),(2, 4), (3, 4)}.

Example2:

let A={1,2,3} and R ={(1,2),(1,3),(3,2)}. Then Ris a relation on Asinceit is a subset of A×A with respect to this relation:

1R2, 1R3, 3R2 but (1,1)∉R&(2,1)∉R

Thedomain ofR is {1,3} and

The rangeofR is {2,3}

Example3:

LetA={1,2,3}.DefinearelationRonAbywriting(x, y)∈R,suchthat a≥b,list the element ofR

$aRb \leftrightarrow a\geq b, a,b\in A$

$\therefore$ R={(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)}.

Example4:

A relation on thesetZofintegers is “m divides n.”A common notation forthis relation is to writem|nwhen m divides n. Thus 6|30 but $7 \nmid 25$.

Representation ofrelations:

1) Bylanguage
2) Byordered pairs
3) Byarrow form
4) Bymatrixform

5) Bycoordinates
6) By graphform

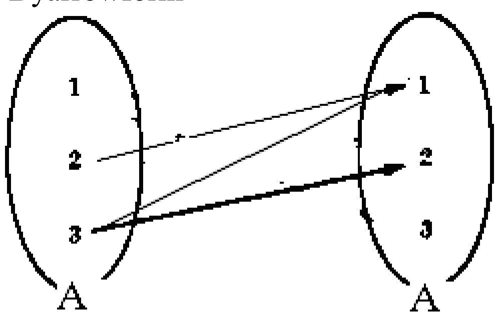
Example:

Let A={1,2,3}, therelation R on Asuch that: aRb $\leftrightarrow$ a>b;   a,b$\in$A

1) Bylanguage:
   R={(a,b): a,b$\in$A and aRb $\leftrightarrow$ a>b}

2) Byordered pairs
   R ={(2,1),(3,1),(3,2)}

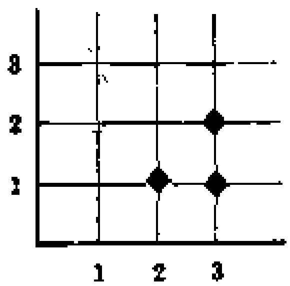
3) Byarrowform



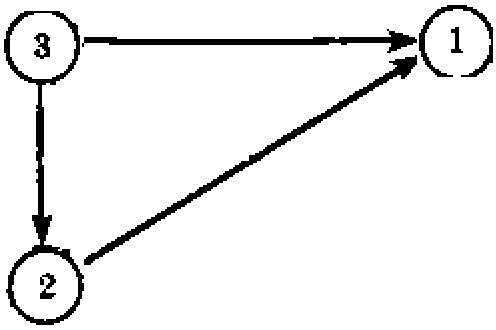
4) Bymatrixform

	1	2	3
1	0	0	0
2	1	0	0
3	1	1	0

5) Bycoordinates



6) By graphform



TYPESOF RELATIONS:

Properties ofrelations:

Let R bea relation on theset A

1) Reflexive:Risreflexiveif: $\forall a \in A \rightarrow aRa$or (a,a) $\in$R; $\forall$a,b $\in$A..ThusR isnot reflexiveifthereexists a$\in$Asuch that (a,a)$\notin$R.

2) Symmetric: aRb $\rightarrow$bRa $\forall$a,b$\in$A. ifwhenever *(a, b)*$\in$ *R*then*(b, a)*$\in$*R*. Thus*R*is not symmetricifthere exists *a, b* $\in$ *A*such that*(a, b)*$\in$ *R*but*(b, a)*$\notin$*R*.

3) Transitive: aRb$\wedge$bRc$\rightarrow$aRc.thatis,ifwhenever *(a, b),(b,c)*$\in$*R* then *(a,c)*$\in$ *R*. Thus *R*is not transitiveifthereexist *a, b, c*$\in$ *R*such that *(a, b)*,*(b, c)*$\in$ *R*but*(a, c)*$\notin$*R.*

4) Equivalence relation :it is Reflexive&Symmetric&Transitive. That is,*R*is an equivalence relation on *S*ifit has the followingthreeproperties:

a-For every$a \in S$, *aRa*. b-
If*aRb*, then*bRa*.
c-If*aRb*and *bRc*, then *aRc*.

5) Irreflexive: $\forall a \in A (a,a) \notin R$
6) AntiSymmetric:ifaRbandbRa$\rightarrow$a=b
therelations$\geq$,$\leq$and$\subseteq$areantisymmetric

Example5: Considertherelation ofC ofset inclusion on anycollection ofsets:
1) A$\subset$Aforanyset,so $\subset$is reflexive
2) A$\subset$Bdosenot implyB $\subset$A,so $\subset$is notsymmetric
3) If A$\subset$BandB$\subset$C thenA$\subset$C, so $\subset$is transitive
4) $\subset$is reflexive, notsymmetric&transitive, so$\subset$isnot equivalencerelations
5) A$\subset$A,so $\subset$is notIrreflexive
6) If A$\subset$BandB$\subset$AthenA=B,so$\subset$is anti-symmetric

Example6:IfA={1,2,3} and R={(1,1),(1,2),(2,1),(2,3)}
Is R equivalence relation ?
1) 2is in Abut (2,2)$\notin$R,so R is not reflexive
2) (2,3) $\in$R but (3,2)$\notin$R,so R isnot symmetric
3) (1,2) $\in$R and (2,3)$\in$R but (1,3)$\notin$R, soR is nottransitive So
R is not Equivalence relation

Example7 :What is theproperties oftherelation=?

1) a=afor anyelement $a\in A$, so =is reflexive
2) Ifa=b thenb = a, so=is symmetric
3) Ifa=band b =cthen a= c, so =is transitive
4) =is (reflexive+symmetric+transitive), so =is equivalence
5) a= a, so=is notIrreflexive
6) Ifa=band b =athen a=b, so =is anti-symmetric

Remark: Thepropertiesofbeingsymmetric and being antisymmetricarenot negatives ofeach other.For example, therelation $R=\{(1, 3), (3, 1),(2, 3)\}$ is neither symmetricnorantisymmetric. On theotherhand,the relation $R=\{(1, 1), (2, 2)\}$ is both symmetric andantisymmetric.

-ReflexiveClosures

Let Rbearelation on aset A. Then:

$R\cup\{(a,a)|a\in A\}$ isthe reflexive closure of$R$.Inother words,**reflexive(*R*)** is obtained bysimplyaddingto Rthose elements(a,a)in thediagonal whichdo not alreadybelongto R.

-SymmetricClosures

$R \cup R^{-1}$isthe symmetric closure of$R$.inother words,**symmetric(*R*)** isobtainedby addingto Rall pairs(b, a)whenever(a, b)belongs to R.

EXAMPLE : Considerthe relation $R=\{(1, 1),(1, 3),(2, 4),(3, 1),(3, 3),(4, 3)\}$ on theset $A=\{1, 2, 3, 4\}$.Then

reflexive$(R)=R\cup\{(2,2),(4,4)\}$and

symmetric$(R)=R\cup\{(4,2),(3,4)\}$

-TransitiveClosure

$R^*$is thetransitive closureof$R$, *where:*

$R^*=R\cup R^2\cup R^3\cup\ldots\cup R^n$ and$R^2=R\circ R$ and $R^n=R^{n-1}\circ R$

Theorm: Suppose$A$is afiniteset with n elements andLet Rbearelationon aset A with n elements. Then : transitive$(R) =R\cup R^2\cup R^3\cup\ldots\cup R^n$

EXAMPLE: Considertherelation $R=\{(1,2),(2,3),(3, 3)\}$on $A=\{1,2,3\}$. Then:

$R2 =R\circ R=\{(1, 3),(2, 3), (3, 3)\}$ and

$R3 =R2\circ R=\{(1, 3),(2,3),(3, 3)\}$ then

transitive$(R)=\{(1, 2),(2, 3),(3, 3),(1, 3)\}$

Inverse relations:

$R^{-1}=\{(b,a): (a,b)\in R\}$

Example1 :

Let R bethe followingrelation on $A =\{1,2,3\}$ $R =\{(1,2),(1,3),(2,3)\}$

$\therefore R^{-1}=\{(2,1),(3,1),(3,2)\}$

ThematrixforR :

$$MR = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$MR^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

MR^{-1} is thetransposeofmatrixR

Composition ofrelations:

Let A,B, C besets and let :

R :A →B (R⊂A×B)

S : B→C (S⊂B×C)

Thereis arelation fromAto C denoted by

R°S(compositionofRandS):A→C

R°S={(a,c):∃b∈B forwhich(a,b) ∈R and (b,c) ∈S}

Example: let A ={1,2,3,4}

B={a, b, c, d}

C ={x,y, z}

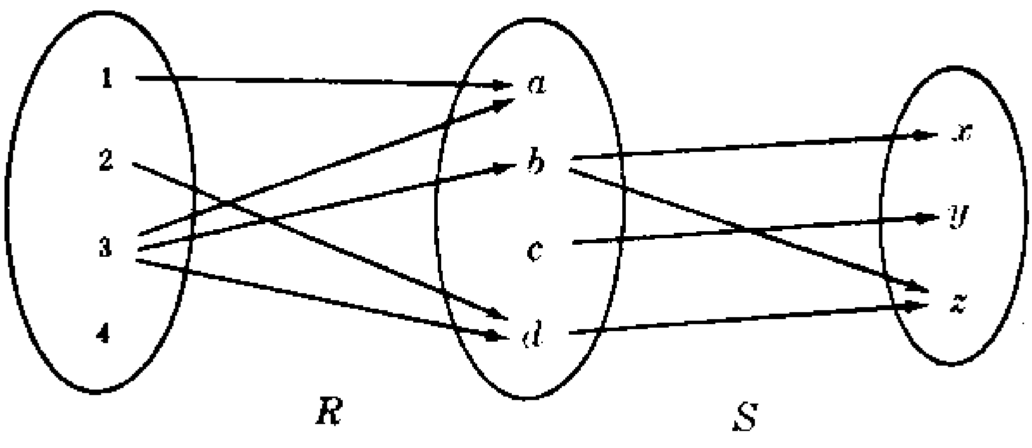
R ={(1,a),(2,d),(3,a),(3,d),(3,b)}

S ={(b,x),(b,z),(c,y),(d,z)}

FindR°S?

Solution :

1) The first waybyarrow form



Thereis anarrow(path)from 2 to d which is followed byan arrowfrom d to z

2Rd and dSz ⇒ 2(R°S)z

and3(R◦S)xand 3(R◦S)z

so R°S= {(3,x),(3,z),(2,z)}

2) Thesecond waybymatrix:

$$MR = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad MS = \begin{array}{c|ccc} & x & y & z \\ \hline a & 0 & 0 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{array}$$

R °S= $M_R.M_S$=

$$\begin{array}{c|ccc} & x & y & z \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 4 & 0 & 0 & 0 \end{array}$$

R°S= {(2,z),(3,x),(3,z)}

Theorem 2.1:Let A,B,C and Dbesets. SupposeR is a relation from AtoB, S is a relation fromBto C, and
Tis a relation from C toD. Then (R
∘S)∘T=R∘(S ∘T)

***n*-ARY RELATIONS**

All the relations discussed above werebinaryrelations. Byan *n-aryrelation*, we mean aset ofordered *n*-tuples. For anyset *S*, asubset oftheproduct set $S^n$is called an *n*-aryrelation on *S*.In particular, asubset of$S^3$is called a*ternaryrelation* on *S*. **EXAMPLE**
(a) Let *L* bealinein theplane. Then “betweenness”is aternaryrelation *R*on the pointsof*L*; that is, *(a, b, c)*∈*R,* if*b* lies between*a* and *c*on *L*.

(b) The equation $x2 + y2 + z2 = 1$ determines aternaryrelation *T* on theset **R**of real numbers. That is, atriple*(x, y, z)*belongs to *T* if*(x, y, z)*satisfies theequation, which means*(x,y, z)*is the coordinates ofapoint in **R**3 on thesphere*S* with radius 1 and centerat theorigin *O*=*(*0, 0, 0*)*.

Homework:

1) Considerthe followingrelations on theset *A*={1, 2, 3}:

R={(1, 1),(1, 2),(1, 3),(3, 3)},
S ={(1, 1)(1, 2),(2,1)(2, 2),(3, 3)},
T ={(1, 1),(1, 2),(2, 2),(2, 3)}
∅=emptyrelation
A×*A*=universal relation

Determine whetherornot each oftheaboverelations on $A$is:
(a) reflexive; (b)symmetric; (c)transitive; (d)antisymmetric.

2) fortherelation $R=\{(a, a),(a, b),(b,c),(c, c)\}$on theset $A=\{a, b, c\}$.
Find: (a)reflexive(R);(b)symmetric(R);(c)transitive(R).

Function:

Function is an importantclass of relation.

Definition:

Let A,Bbe two nonemptysets,afunction F: A→Bis a rule whichassociates with **each** element ofAa**unique**element in B.

Theset Ais called the**domain** ofthefunction, and theset Bis called the**range**ofthe function.

Example1:

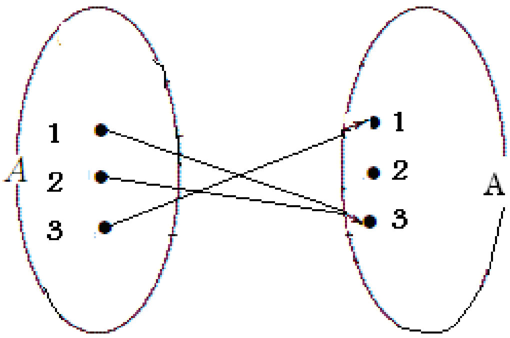
Considerthefunction$f(x)=x^3$,i.e.,$f$assignstoeachrealnumberitscube.Thentheimage of2is8,andso we maywrite$f(2)=8$.

Example2 :

considerthe followingrelation on thesetA={1,2,3}

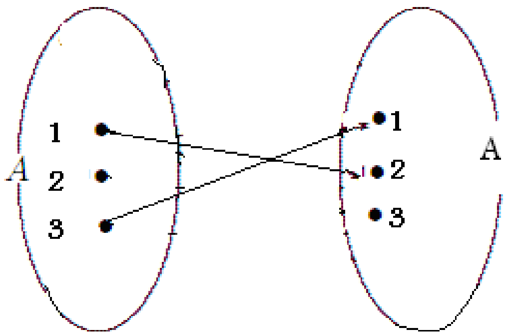
F={(1,3),(2,3),(3,1)}

Fis a function



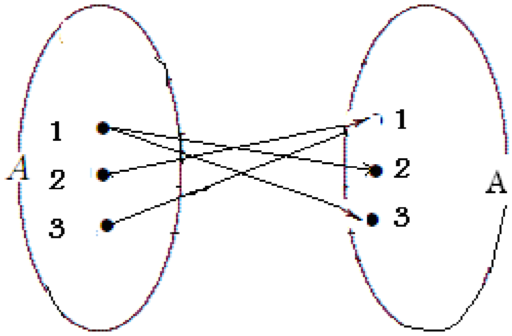
G={1,2},(3,1)}

Gis not a function fromAto A



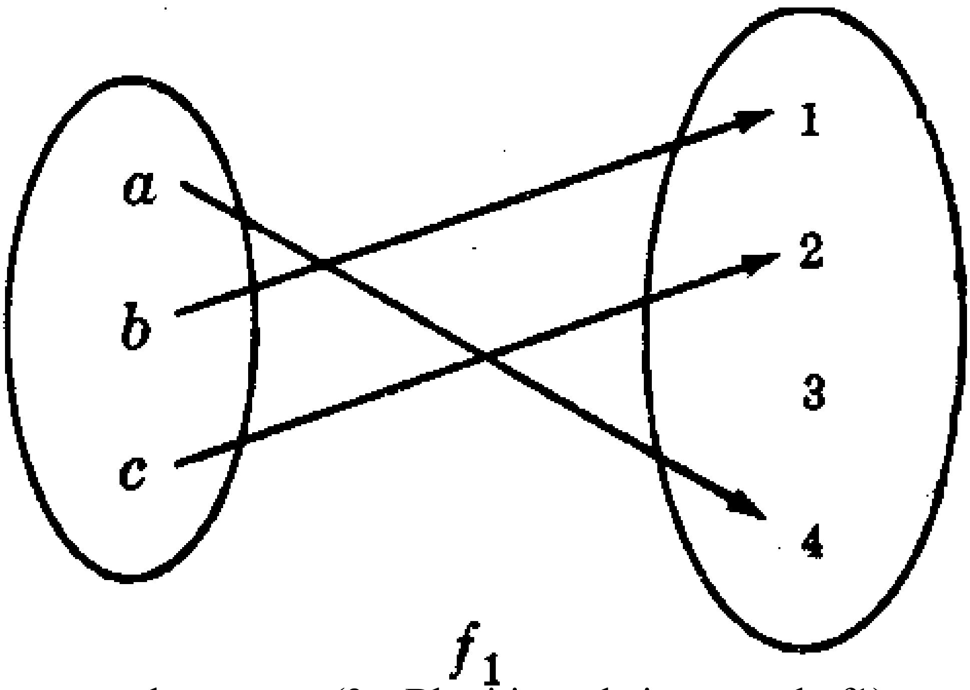
H={(1,3),(2,1),(1,2),(3,1)}

His not a function

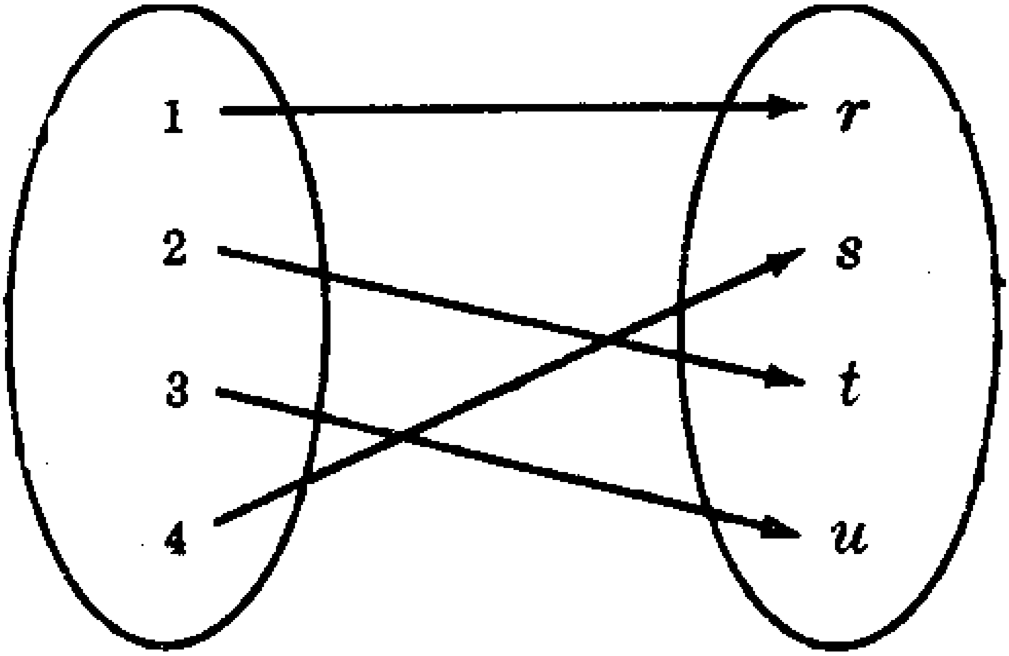


One-to-one ,onto andinvertiblefunctions :

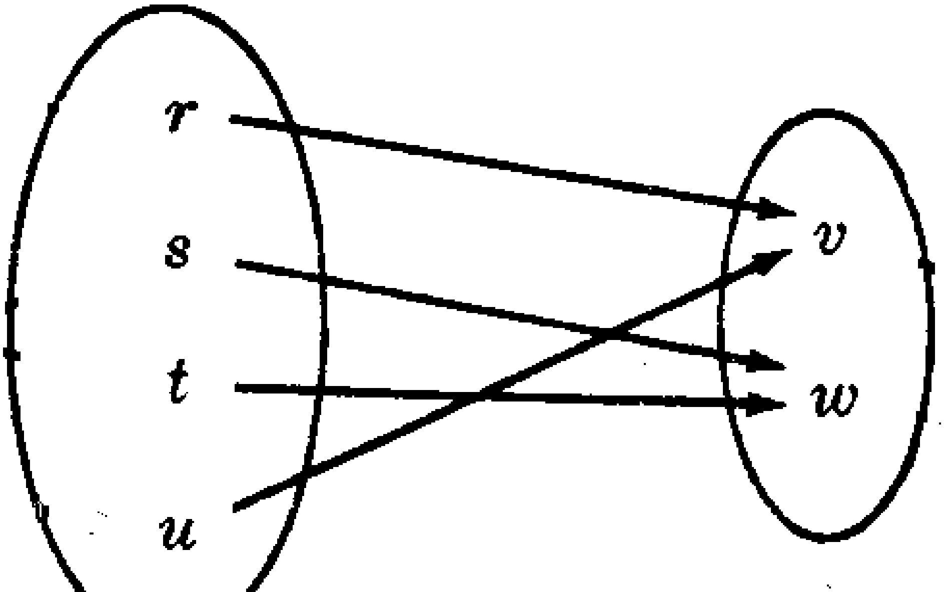
1) One–to-one : afunction$F:A\rightarrow B$is saidto be one-to-oneifdifferentelements in thedomain (A)havedistinct images.
   Or If $F(a)=F(a')$ $\Rightarrow$ $a=a'$
2) Onto : $F:A\rightarrow B$is said tobe anonto function if each element ofBistheimageof some element of A.
   $\forall b\in B$ $\exists$ $a\in A$**:**$F(a)=b$
3) Invertible (One-to-onecorrespondence)
   $F:A\rightarrow B$isinvertible if itsinverse relationf^{-1}isa function$F:B\rightarrow A$
   $F:A\rightarrow B$is invertible ifandonlyifFis**both** one-to-one and onto
   $F^{-1}:\{(b,a)\forall(a,b)\in F\}$



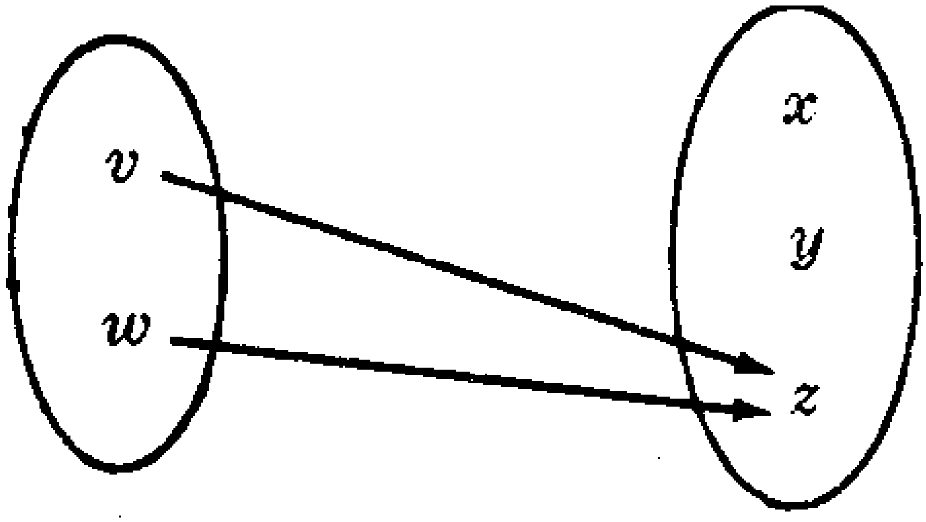
onetoonebutnotonto($3\in B$butitisnottheimageunderf1)



bothonetoone&onto
(oronetoonecorrespondencebetweenAandB)



f_3
notonetoone&onto



notonetoone&notonto

Graphofa function:

By*areal polynomial function*, wemeana function *f*: **R**→ **R**ofthe form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

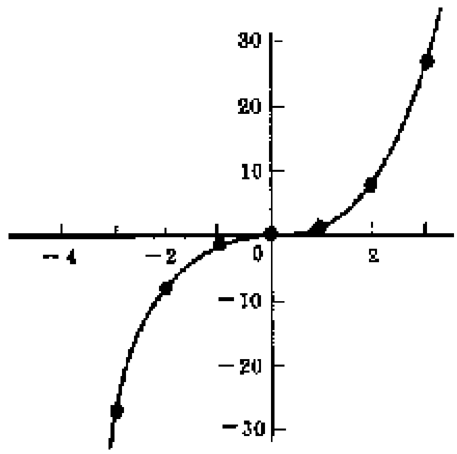
wherethe$a_i$arereal numbers. Since**R**is an infiniteset, it would beimpossibleto plot each point ofthegraph.However, thegraph ofsuch a function can beapproximated by first plottingsomeofits points and then drawing asmooth curvethough thesepoints. The tablepoints areusuallyobtained from atable wherevarious values are assigned to xand the correspondingvalueof f(x) computed.

Example1 : let f:R→R and f(x)=x^3, find f(x)

f(3) =3^3=27

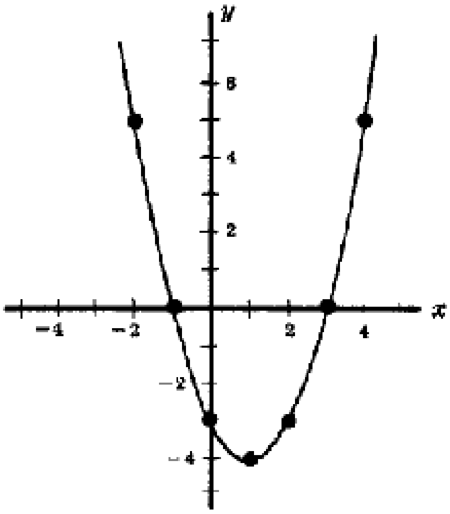
f(-2) =$(-2)^3$= -8

x	$f(x)$
−3	−27
· 2	8
1	−1
0	0
1	1
2	8
3	27



Example2: let f: R→R and $f(x)=x^2-2x-3$,, find f(x)

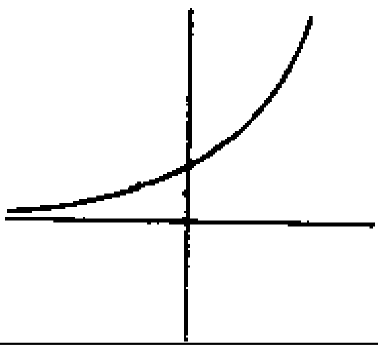
x	$f(x)$
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5



Geometrical CharacterizationofOne-to-OneandOnto Functions

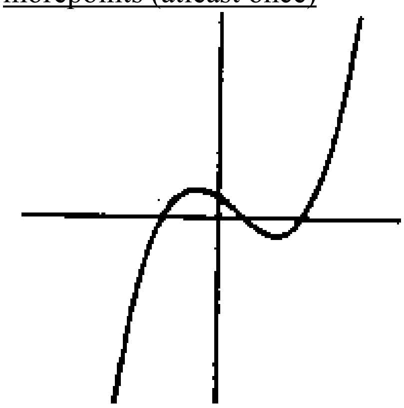
Forthe functions oftheform f: R →R. thegraphs ofsuch functions maybeplotted in theCartesian planeand functions maybeidentified with theirgraphs, so the concepts of beingone-to-oneand onto havesomegeometricalmeaning:

(1) f:R →R is said to be<u>one-to-one</u>ifthere areno 2 distinct pairs (a_1,b)and (a_2,b)in the graph one-to-oneorifeach horizontal lineintersects thegraph offin <u>at most onepoint.</u>



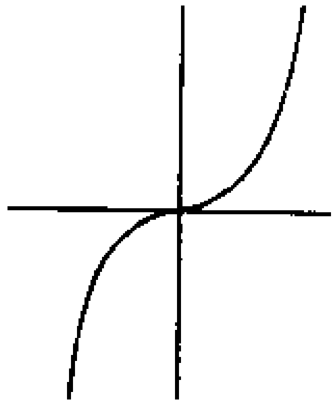
$f_2(x) = 2^x$

(2) f:R →R is an <u>onto</u> function if each horizontal lineintersects thegraphoff<u>at oneor morepoints (atleast once)</u>

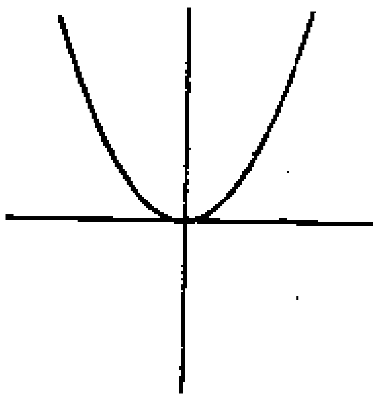


$f_3(x) = x^3 - 2x^2 - 5x + 6$

(3) iffis both one-to-one and onto, i.e. invertible,then each horizontal line will intersect thegraph offatexactlyonepoint.



$f_4(x) = x^3$



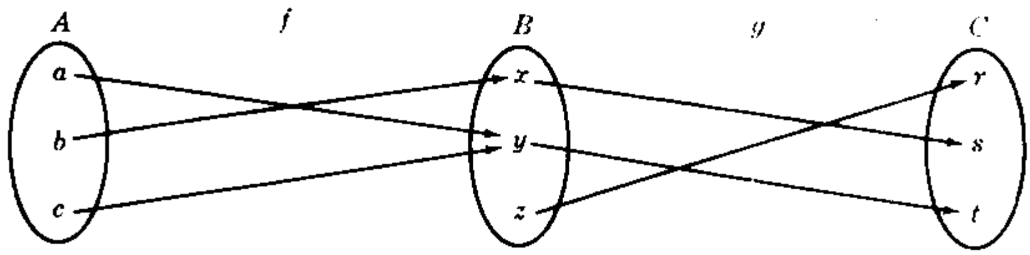
$f_1(x) = x^2$

f(x) NOT (ONE-TO-ONE)& NOT (ONTO)

Compositionoffunction:

Letf:A→B andg:B→C,tofind the composition functiongof:A→C

(gof)(a)=g(f(a))=g(y)=t
(gof)(b)=g(f(b))=g(x)=s
(gof)(c)=g(f(c))=g(y)=t



SEQUENCES OFSETS

A*sequence*is a function from theset **N**={1, 2, 3,. . .} ofpositiveintegersinto aset A. Thenotation $a_n$is used to denotetheimageoftheinteger$n$. Thus asequenceis usually denoted by

$$a1, a2, a3, \ldots$$

A finitesequenceoveraset Ais a function from {1, 2, . . . , m} into A,Such a finite sequenceiscalled a*list.*

EXAMPLE

(a) The followingaretwo familiarsequences:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \text{ which may be defined by } a_n = \frac{1}{n};$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \text{ which may be defined by } b_n = 2^{-n}$$

Notethat the first sequencebegins with n =1 and thesecond sequencebegins with n =0.

(b) Thesequence1,−1, 1,−1, . . . maybedefined by $an = (-1)^n$
, wherethesequencebegins with n =0.

Summation Symbol, Sums

Hereweintroduce thesummationsymbol∑(theGreeklettersigma).Considera sequence$a1$,$a2$, $a3$, ...Then wedefinethe following:

$$\sum_{J=1}^{n} aj = a1 + a2 + \cdot\cdot\cdot + an$$

EXAMPLE :

$$\sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4+9+16+25 = 54$$

$$\sum_{j=1}^{n} j = 1 + 2 + \cdot\cdot + n = n(n+1)/2,$$ forexample, 1 +2+ · · +50= (50 x51)/2=1275

RECURSIVELY DEFINEDFUNCTIONS

A function is said to be*recursivelydefined* ifthefunction definition refers to itself.In order forthedefinition not to be circular, the function definition must havethe following two properties:

(1) Theremust be certainarguments, called *basevalues*, for which thefunction does not referto itself.

(2) Each timethefunction does referto itself, theargument ofthe function must be closer to abasevalue.

A recursive function with thesetwo properties is said to be*well-defined*.

FactorialFunction

Theproduct ofthepositiveintegers from 1 to n, inclusive, is called "n factorial" and is usuallydenoted by$n!$. That is,

$$n! = n(n-1)(n-2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$$

where$0!=1$, so that thefunction is defined forall nonnegativeintegers. Thus:

$0!=1,$ $1!=1,$

$2!=2.1=2,$ $3!=3.2.1=6,$

$4!=4.3.2.1=24$ $5!=5.4.3.2.1=120$

$6!=6.5.4.3.2.1=720$

This is true for everypositiveinteger$n$; that is,

$$n! = n \cdot (n-1)!$$

Accordingly, thefactorial function mayalso bedefined as follows:

Definition ofFactorialFunction:

(a) If$n=0$, then $n!=1$.

(b) If$n>0$, then $n!=n \cdot (n-1)!$

Thedefinition of$n!$is recursive, sinceit refers to itself when it uses$(n-1)!$. However:

(1) Thevalueof$n!$is explicitlygiven when $n=0$ (thus 0 is abasevalue).

(2) Thevalueof$n!$forarbitrary$n$ is defined in terms ofasmallervalueof$n$ which is closerto thebasevalue0.

Accordingly, thedefinition is not circular, or, in other words, the function is well-defined.

EXAMPLE :the4!Canbe calculated in 9 stepsusingtherecursivedefinition .

(1) $4! = 4 \cdot 3!$

(2) $3! = 3 \cdot 2!$

(3) $2! = 2 \cdot 1!$

(4) $1! = 1 \cdot 0!$

(5) $0! = 1$

(6) $1! = 1 \cdot 1 = 1$

(7) $2! = 2 \cdot 1 = 2$

(8) $3! = 3 \cdot 2 = 6$

(9) $4! = 4 \cdot 6 = 24$

Fibonacci Sequence

TheFibonacci sequence(usuallydenoted by$F_0, F_1, F_2, \ldots$)is as follows:

$$0, \ 1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ 34, \ 55, \ \ldots$$

That is, $F_0=0$ and$F_1=1$ and each succeedingterm is thesum ofthetwo preceding terms. Forexample, thenext two terms ofthesequenceare

$34+55=89$ and $55+89=144$

Fibonacci Sequence canbedefined:

(a) Ifn =0, orn=1, thenFn =n.
(b) Ifn>1, then $Fn=F_{n-2}+F_{n-1}$.

Where: Thebasevaluesare0 and 1,and thevalueofFn is defined in terms ofsmaller values ofn whicharecloserto thebasevalues.
Accordingly, this function is well-defined.

Logicandpropositional calculus

A*proposition* (or*statement*)is adeclarativestatement which is trueorfalse, but not both.
Example:the followingsixsentences:
(1) Ice floats in water.
(2) Chinais in Europe.
(3)2 +2 =4
(4)2 +2 =5
(5) Whereareyougoing?
(6) Doyourhomework.
The first fourarepropositions, thelast two arenot. Also, (1) and (3) aretrue, but (2) and(4)are false.

CompoundPropositions

It is theproposition thatcomposed ofsubpropositions and various connectives.
Primitiveproposition istheproposition that cannot bebroken down into simplerpropositions. For example, the abovepropositions areprimitivepropositions, while:
"Roses arered and violets areblue."and
"John is smart orhestudies everynight.",Arecompound.

BASICLOGICAL OPERATIONS

1-Conjunction, $p \wedge q$
2-Disjunction, $p \vee q$
3-Negation, $\neg p$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	$\neg p$
T	F
F	T

(c) "not p"

EXAMPLE: Considerthefollowingfourstatements:
(i) Icefloats in water**and** 2 +2 =4.
(ii) Icefloats in water**and** 2 +2 =5.
(iii) Chinais in Europe**and** 2 +2 =4.
(iv) Chinais in Europe**and** 2 +2 =5.
Onlythefirst statement is true. Each oftheothersis falsesince at least oneofits substatements is false.

EXAMPLE: Considerthefollowingfourstatements:
(i) Icefloats in water**or**2 +2 =4.
(ii) Icefloats in water**or**2 +2 =5.
(iii) Chinais in Europe**or**2 +2 =4.
(iv) Chinais in Europe**or**2 +2 =5.

Onlythelast statement (iv)is false. Each oftheothers is truesinceat leastoneofits sub-statements is true.

EXAMPLE: Considerthefollowingsix statements:

(a1)Ice floats in water. (a2)It is falsethat ice floats in water. (a3)Icedoes not float in water.

(b1)2 +2 =5 (b2)It is falsethat 2+2=5. (b3)2 +2≠5

Then (a2) and(a3)areeach thenegation of(a1); and (b2) and (b3) are eachthenegation of (b1). Since (a1)is true, (a2) and (a3) are false;and since (b1)is false,(b2)and (b3) aretrue.

PROPOSITIONS ANDTRUTH TABLES

Thetruth table forthe compoundproposition ¬(p∧¬q)is:

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

TAUTOLOGIES AND CONTRADICTIONS

Somepropositions P(p,q, . . .) contain onlyTin thelast column oftheirtruth tables or, in other words, theyaretrueforanytruth values oftheirvariables. Such propositions arecalled tautologies. Analogously, aproposition P(p, q, . . .)is called acontradictionifit contains onlyF in thelast column ofits truth tableor, in other words, ifit is false for anytruth values ofits variables.For example, theproposition “p ornot p,”that is, p ∨¬p, is atautology, and the proposition “p and not p,”that is, p∧¬p, is a contradiction.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(a) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(b) $p \wedge \neg p$

Notethat thenegation ofatautologyis acontradiction sinceit is always false, and thenegation ofa contradiction is atautologysinceit is always true.

LOGICAL EQUIVALENCE

Two propositions P(p, q, . . .) and Q(p, q, . . .) aresaid to belogicallyequivalent, or equal, denoted byP(p, q, . . .)≡Q(p, q, . . .)iftheyhaveidentical truth tables.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$(a)\ \neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$(b)\ \neg p \vee \neg q$

for example, thetruth tables of$\neg$(p ∧q) and $\neg$p∨$\neg$q, both truth tables arethesame, that is, both propositions are falsein the first case and truein theotherthree cases.Accordingly, we can write: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

ALGEBRAOF PROPOSITIONS

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv F$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

CONDITIONAL ANDBICONDITIONAL STATEMENTS

Manystatements, particularlyin mathematics, areofthe form "Ifp then q."Such statements are called conditional statements and aredenoted by: $p \rightarrow q$

The conditional $p \rightarrow q$ is frequentlyread "p implies q"or "p onlyifq."

Another common statement is ofthe form "p ifand onlyifq."Such statements arecalled biconditional statementsandare denotedby $p \leftrightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$(a)\ p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$(b)\ p \leftrightarrow q$

(a) Theconditional p →q is falseonlywhen thefirst part p is true and thesecond part q is false. Accordingly, when p is false, the conditional p →q is true regardless ofthetruth valueofq.
(b) Thebiconditionalp↔q istrue whenever pand qhavethe sametruthvaluesand false otherwise.
Notethat thetruth tableofp →q and ¬p ∨q and areidentical, that is, theyarebothfalseonly in thesecond case. Accordingly, p →q is logicallyequivalent to ¬p ∨q; that is, p →q ≡¬p ∨q

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(c) $\neg p \vee q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$