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قســـم الريـاضيـات

حول المقاسات شبة-الأغمارية رئيسية والمقاسات نصف-الأغمارية

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مقدمه الى كلية العلوم- جامعة بغداد
كجزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
علي كريم كاظم
٢٠٠٥

بِسْمِ ٱللَّهِ ٱلرَّحْمَنِ ٱلرَّحِيمِ

"وَمِنَ ٱلنَّاسِ مَنْ يَشْرِي نَفْسَهُ ٱبْتِغَاءَ
مَرْضَاتِ ٱللَّهِ وَٱللَّهُ رَءُوفٌ بِٱلْعِبَادِ"

صَدَقَ ٱللَّهُ ٱلْعَظِيمْ

البقرة/ آية ٢٠٨

الاهداء

الى هادي الامة ورسول الرحمة محمد بن عبد الله (ص)

الى والدي العطوفين

الى اخوتي الاعزاء

الى اختي الغالية

اهدي هذا الجهد المتواضع

شكر وتقدير

الحمد لله رب العالمين والصلاة والسلام على أشرف الأنبياء والمرسلين الرسول الأمين محمد واله وصحبه ألطاهرين

أتقدم بالشكر الجزيل الى رائد الفكر المعاصر الأستاذ الدكتور **عادل غسان نعوم** وأشكر الدكتوره **وسن خالد** لوقوفهما معي في مرحلة اعداد البحث
ولا أنسى ان أذكر أصدقائي الذين ساعدوني في مرحلة اعداد البحث وبالأخص أخي وصديقي **محمد علي مراد النداوي .**

والله ولي التوفيق

I certify that this thesis was prepared under my supervision at the University of Baghdad as partial requirements of the degree of Master of Science in Mathematics.

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We certify that we have read this thesis and as examining committee examined the student in its context and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics with (   ) standing .

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Dean of the College of Science

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المستخلص

ليكن M مقاسا على R,S حلقة التشاكلات للمقاس M على R .المقاس M_R يسمى رئيسي شبه – أغماري أذا كان لكل تشاكل مقاسي على R من أي مقاس جزئي دوري منM الى M يمكن توسيعه الى حلقة التشاكلات في M .ألمقاس N على R يسمى M – رئيسي أغماري أذا كان لكل تشاكل مقاسي على R من M – مقاس جزئي دوري K من M الى N يمكن توسيعه الى M .المقاس M على R يسمى نصف- أغماري اذا كان M هوM -رئيسي أغماري .

أن هذه المفاهيم درست من قبل نكلسون، يوسف، و ونكوال . الغرض الرئيسي من هذه الافكار هو دراسة المقاسات الرئيسية شبه – أغمارية والمقاسات نصف –اغمارية . سنحاول اعطاء تفاصيل البراهين للنتائج المعروفة ، نورد بعض الامثلة ، ونضيف بعض النتائج الجديدة.

Introduction

Let R be a commutative ring with 1 and M is a unitary right R-module and $S=End_R(M)$. In [15] Nicholson , Park and Yousif studied principally quasi-injective modules where M is called principally quasi-injective module if each R-homomorphism from a principal submodule of M to M can be extended to an endomorphism of M, a ring R is called principally injective if R is a principally quasi-injective R-module [15]. In [21] Wongwal studied M-principally injective modules where an R-module N is called M-principally injective if every R-homomorphism from M-cyclic submodule K of M to N can be extended to M. An R-module M is called semi-injective if it is M-principally injective. In [8] Chotchaisthit asks the following question: for an R-module, when is a quasi- principally-injective module continuous?. An R-module M is called continuous if M has c_1-condition and c_2-condition where M is said to have the c_1-condition if every submodule of M is essential in a direct summand of M [8], and it has c_2-condition if every submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M [15].

The main goal of this thesis is to study principally quasi-injective modules, semi-injective modules and their endomorphisms rings. Further we examine their relations with other known concepts like, local rings ,uniform modules, duo module, self-generators, summand intersection property, summand sum property. We give the details of known results and some examples. We also add few new results (to the best of our knowledge).

The material presented in this thesis is organized in three chapters. Each of chapters one and two is divided into three sections and chapter three consist of two sections.

In section 1, chapter 1, we study different characterization of principally quasi-injective modules (Theorem (1.1.9).

In section 2, we study the Jacobson radical of S and related concepts such as singularity (Theorem (1.2.10). Further we look at the properties of the ideal W(S)={$w \in S \backslash$ 1-βw is monomorphism for all $\beta \in S$}, in principally quasi-injective modules (proposition(1.2.5)).

In section 3, we study further results on principally-injective rings and some notions as weakly injective.

In chapter two we study principally quasi-injective modules and their relations to other classes of modules.

In section 1, we study the relation between principally quasi-injective modules and some properties as a summand intersection property, summand sum property and c_3-condition, these properties can be found in [15], [5], [4].

In section 2, we study uniform submodules. Many of the ideas in this section trace back to camillo [7].

In section 3, we study the relation between principally quasi-injective modules and continuous modules. The main result of this section appeared in (proposition (2.3.1)). We also study duo principally quasi-injective (proposition (2.3.7)).

In chapter three we study semi-injective modules and fully stable modules.

In section 1, we look at the relation between semi-injective modules with π-injectivity and direct injectivity (Theorem (3.1.11)).

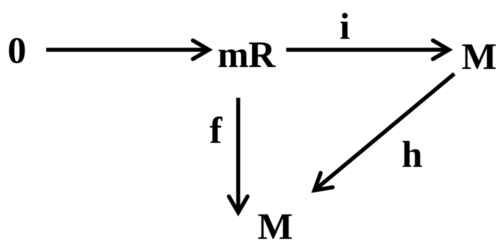
In section 2, we study fully stable and fully invariant modules in principally quasi-injective modules and rings.

CHAPTER ONE

PRINCIPALLY QUASI-INJECTIVE MODULES AND PRINCIPALLY INJECTIVE RINGS

Introduction

Let M be an R-module with endomorphism ring S. As we mentioned in the introduction, an R-module M is called principally quasi-injective module if each R-homomorphism from a principal submodule of M to M can be extended to an endomorphism of M. In other words, the following diagram is commutative:



The ring R is called principally injective if R is principally quasi-injective as an R-module **[14]**.

The concept of principally quasi-injective modules was introduced in 1999.

In this chapter we study principally quasi-injective modules and principally injective rings. We recall the known results about these concepts and we give the details of the proofs of these results, we also add few new results (to the best of our knowledge).

This chapter consists of three sections.

In section 1, we recall the definitions of principally quasi-injective module and principally injective ring. More over, we recall some properties about principally quasi-injective modules.

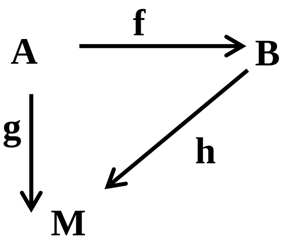
In section 2, we study the Jacobson radical of the ring of endomorphism S of a principally quasi-injective module and its relation with other concepts.

In section 3, we study principally injective rings. Some of the results about these rings are corollaries to corresponding results on principally quasi-injective modules.

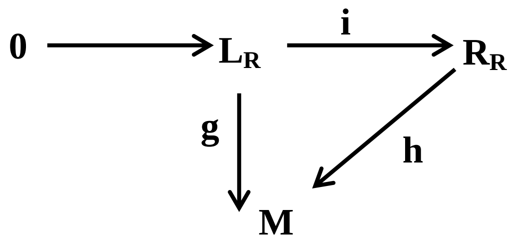
<u>Section 1.1 Principally Quasi-Injective Modules:</u>

In this section we study principally quasi-injective modules and their endomorphism rings. Most of the results of this section appeared in **[14],[15]**. However, we give the details of the proofs and add few new results (to the best of our knowledge).

Recall that an R-module M is injective if given any monomorphism $f : A \to B$ and any homomorphism $g : A \to M$, there exists a homomorphism $h : B \to M$ such that $h \circ f = g$. In other words the following diagram is commutative where A, B are R-modules.

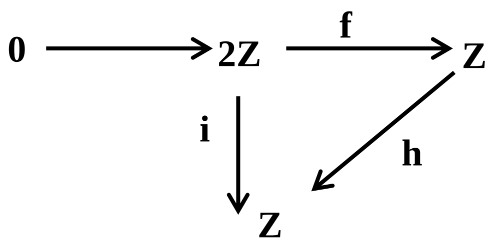


Equivalently, an R-module M is injective if for every ideal L_R of R_R and any homomorphism $g : L \to M$, g can be extended to a homomorphism $h : R \to M$ **[10, p.130]**.



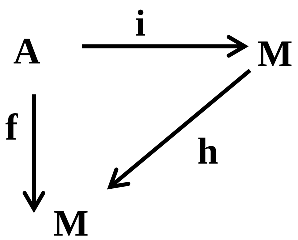
It is well known that Q as a Z-module is injective[11], but Z as a Z-module is not injective module. In fact, let $f : 2Z \to Z$ be defined by $f(2n) = 3n \ \forall n \in Z$ .If there is h: Z $\to$ Z which extends f, $h \circ f = i$,**then** $h(f(2n)) = h(3n) \neq 2n = i(2n)$

In particular, if $n = 1$, then $h(f(2)) = h(3) = 3 \neq 2 = i(2)$. Hence $h \circ f \neq i$



Definition 1.1.1 [6] :

An R-module M is said to be quasi-injective if any homomorphism $f : A \rightarrow M$ where A is a submodule of M, can be exteneded to an endomorphism $h : M \rightarrow M$, i.e., the following diagram is commutative, where i is the inclusion map.



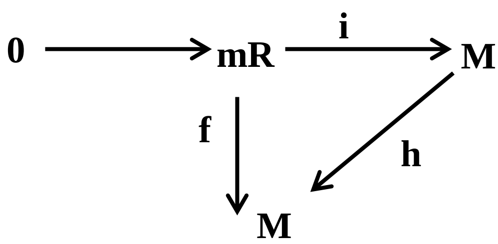
A ring R is called self-injective (quasi–injective) if it is a quasi-injective R-module.

It is clear that every injective module is quasi-injective so as every simple module. An example of quasi-injective Z-module which is not injective Z-module is Z/2Z, it is simple but it is not injective because it is not divisible.

Definition 1.1.2 [15] :

An R-module M is called principally quasi-injective if each R-homomorphism from cyclic submodule of M to M can be extended to an endomorphism of M, i.e., the following diagram is commutative,

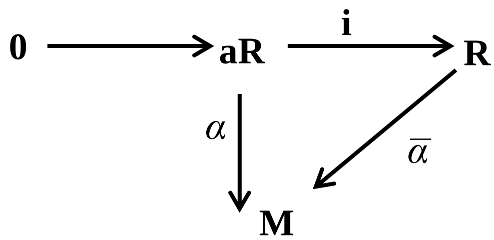
$h \circ i = f$



Note 1.1.3: We use the notation P.Q.-injective for principally quasi-injective.

Definition 1.1.4 [14] :

An R-module M is called principally injectuive if each R-homomorphism $\alpha : aR \to M$ such that $a \in R$, extends to R , i.e., the following diagram is commutative, $\overline{\alpha} \circ i = \alpha$.



Remarks and Examples 1.1.5 :

(1) It is clear that every injective module is principally injective.

(2) If every cyclic submodule of M is a summand then M is P.Q.-injective module, in fact, $xR \leq M$, xR is a direct summand of M, there exists $B \leq M$ such that $M = xR \oplus B$. Now let $\alpha : xR \to M$ be a homomorphism. Define $\overline{\alpha} : xR \oplus B \to xR \oplus M$ by $\overline{\alpha}(xr, y) = \alpha(xr)$, it is clear that $\overline{\alpha}$ is an extension of α.

(3) Recall that a module M is called Z-regular if every cyclic submodule is a projective and direct summand **[16]**. Thus every Z-regular module is P.Q.-injective module.

(4) The ring R is called principally injective if R is a P.Q.-injective R-module **[15]**. Hence every (von Neuman) regular ring R which is not quasi-injective is an example of a P.Q.-injective module that is not quasi-injective.

Note 1.1.6: We will use P-injective for principally injective

Remark 1.1.7: Let S=end$_R$ (M)=the ring of R-endomorphisms of M. If M is a right R-module, then M can be made into a left S-module as follows.

Define $\Phi : SxM \to M$ by $\Phi(f,m) = f(m)$, then

(1) $(f_1 + f_2)(m) = f_1(m) + f_2(m)$

(2) $f(m+n) = f(m) + f(n)$

(3) $(f_1 f_2)(m) = f_1(f_2(m))$, *where* $f, f_1, f_2 \in S$, $m, n \in M$

Note 1.1.8: Let M be an R-module, we fix some notation:

(1) $ann_M(r) = \{m \in M \setminus mr = 0\}$.

(2) $ann_R(m) = \{r \in R \setminus mr = 0\}$.

(3) $Sm = \{f(m) \setminus f \in S\}$.

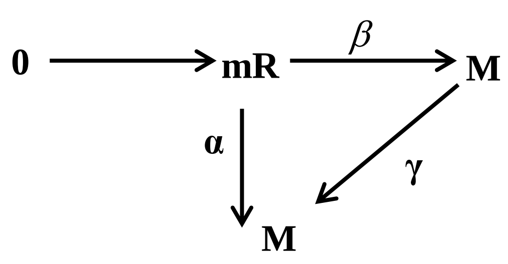
Theorem 1.1.9:[15]: Given a module M$_R$ with S=end(M$_R$), the following are equivalent.

(1) $\forall m \in M$, every R-homomorphism $mR \to M$ can be extended to an endomorphism in S, i.e., M is P.Q.-injective module.

(2) $ann_M\, ann_R(m) = Sm\ \forall m \in M$.

(3) If $ann_R(m) \subseteq ann_R(n)$ where $m, n \in M$ then $Sn \subseteq Sm$.

(4) $\forall m \in M$, if the R-homomorphism $\alpha, \beta : mR \to M$ are given with β is monomorphism, then there exists $\gamma : M \to M$ such that $\gamma \circ \beta = \alpha$, i.e., the following diagram is commutative:



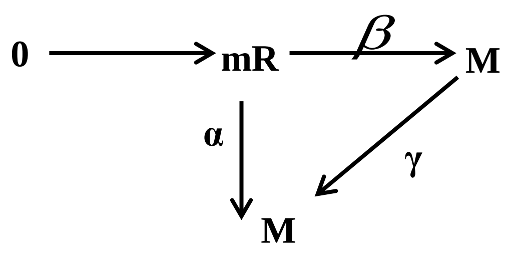
Proof. 1⇒2 Let $f(m) \in Sm$ where $f \in S$. If $mr = 0$, then $0 = f(mr) = f(m)r$. This implies that $f(m) \in ann_M\, ann_R(m)$, hence $Sm \subseteq ann_M ann_R(m)$. To show the opposite inclusion, let $n \in ann_M ann_R(m)$. Define $\gamma : mR \to M$ by $\gamma(mr) = nr \,\forall r \in R$. γ is well-defined, in fact, let $mr_1 = mr_2$ $r_1, r_2 \in R$, $mr_1 - mr_2 = m(r_1 - r_2) = 0$, then $r_1 - r_2 \in ann_R(m)$, hence $n(r_1 - r_2) = 0$, $nr_1 - nr_2 = 0$, implies that $nr_1 = nr_2$. By (1) γ extends to $\bar{\gamma} \in S$.

Now n=γ(m)=($\bar{\gamma}\circ$i) (m)= $\bar{\gamma}$[i(m)]= $\bar{\gamma}$(m) $\in Sm$. Hence $ann_M\,(ann_R(m)) \subseteq Sm$. This proves(2)

2 ⇒3 Let $f(n) \in Sn$. By (2) $Sn = ann_M ann_R(n)$, then $f(n) \in ann_M ann_R(n)$. Since $ann_R(m) \subseteq ann_R(n)$, then $ann_M\,(ann_R(m)) \subseteq ann_R(n)) = ann_M ann_R(n) \subseteq ann_M ann_R(m)$. This implies that $f(n) \in ann_M ann_R(m)$. Hence by (2) $f(n) \in Sm = ann_M ann_R(m)$. This means $Sn \subseteq Sm$.

3⇒4 Since β is monomorphism, we have $ann_R(\beta m) \subseteq ann_R(\alpha m)$, in fact, let $r \in ann_R(\beta m)$, then $\beta(m)r = \beta(mr) = 0$. Thus $mr \in \ker\beta$ hence $mr = 0$, so $\alpha(mr) = \alpha(m)r = 0$. Which implies $r \in ann_R(\alpha m)$, so $ann_R(\beta m) \subseteq ann_R(\alpha m)$. By (3)$S\alpha(m) \subseteq S\beta(m)$. Then there exists $\gamma \in S$ such that $\alpha(m) = \gamma[\beta(m)]$ as required.

4 ⇒1 Take $\beta : mR \to M$ be the inclusion in (4). Then by (4) there exists $\gamma : M \to M$ such that the following diagram is commutative. Hence $\alpha : mR \to M$ extends to an endomorphism in S. This means proving (1).



Proposition 1.1.10 [15] : Let M be an R-module, then for each $m \in M, \alpha \in S, S\alpha + ann_S(m) \subseteq ann_s[\ker(\alpha) \cap mR]$.

Equality holds if M is P.Q.-injective module.

Proof. Suppose that $\beta \in ann_S[\ker(\alpha) \cap mR]$

Claim $ann_R(\alpha m) \subseteq ann_R(\beta m)$, in fact, let $r \in ann_R(\alpha m)$, i.e., $\alpha(mr)$ =0 = $\alpha(m)r$. Therefore, $mr \in$ ker(α) $\cap$mR.. Since $\beta \in ann_S$ [*ker* (α) $\cap$ *mR*], then $\beta(mr) = \beta(m)r = 0$. Hence $r \in ann_R(\beta m)$. Since M is P.Q.-injective, then by theorem (1.1.9(3)) $\beta(m) \in S\alpha(m)$.Say $\beta(m) = \gamma\alpha(m)$ where $\gamma \in S$, so $\beta(m) - \gamma\alpha(m) = (\beta - \gamma\alpha)(m) = 0$, hence $\beta - \gamma\alpha \in ann_S(m)$, thus $\beta \in S\alpha + ann_S(m)$. This means ann_S[*ker* (α) $\cap$ *mR*] $\subseteq S\alpha + ann_S(m)$.

Before the next result we need the following definition **[15]**. An R-module M is said to be principally self-generator if for every element m$\in$M, there exists an epimorphism $\lambda: M_R \to mR$, i.e., there exists $m_1 \in$M such that $\lambda(m_1)=m$.

For example, every cyclic module is principally self-generator,in particular R_R is principally self-generator R-module. More over every Z-regular module is principally self-generator but Q is not[16].

Proposition 1.1.11 [15] : If M is a principal module which is principally self-generator with S=end (M_R), then the following conditions are equivalent.

(1) M_R is P.Q.-injective module .

(2) $ann_S[ker(\alpha)\cap mR]=S\alpha+ann_S(m)$ for all $\alpha\in S$ and $m\in$M

(3) $ann_S[ker(\alpha)]=S\alpha$ for all $\alpha\in S$.

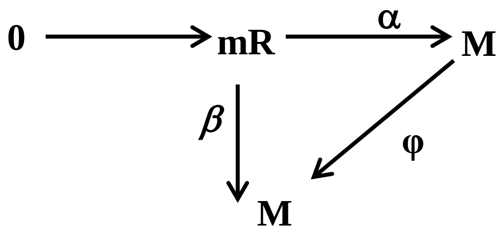
(4) $ker(\alpha)\subseteq ker(\beta)$ where α, $\beta\in S$, implies that $\beta\in S\alpha$.

Proof. 1⇒2: This follows from proposition (1.1.10)

2⇒3 If $M=m_oR$, take $m=m_o$ in (2), i.e., $ann_S[ker(\alpha)\cap m_oR]=S\alpha+ann_S(m_o)$. Let $f\in ann_S[ker(\alpha)\cap m_oR]$, then $f(x)=0$ where $x\in ker(\alpha)$ and $x=m_or$ where $r\in R$, $f(x)=f(m_or)=f(m_o)r=0$, implies that $r=0$. This gives $ann_S[ker(\alpha)]=S\alpha$.

3⇒4 This is because $ann_S[ker(\alpha)]=\{\beta\in S\backslash ker(\alpha)\subseteq ker(\beta)\}$, in fact , $\beta[\alpha(m)]=0$, implies $\alpha(m)\in ker(\beta)$. But $\alpha(m)=0$, hence $m\in ker(\alpha)\subseteq ker(\beta)$,i.e., $ann_S[ker(\alpha)]=\{\beta\in S\backslash\ \beta[\alpha(m)]=0\}$. By (3), $S\alpha=ann_S[ker(\alpha)]$, implies that $\beta\in S\alpha$.

4⇒1 Let $\gamma:mR\to M$ be R-homomorphism where $m\in M$. Now we will take $M=m_oR$ and choose α, β in S such that $m=\alpha(m_o)$ and $\gamma(m)=\beta(m_o)$, we claim that $ker(\alpha)\subseteq ker(\beta)$, in fact, if $k\in ker(\alpha)$, write $k=m_or$ such that $k\in M$, $r\in R$. Now $\beta(k)=\beta(m_or)=\beta(m_o)r=\gamma(m)r=\gamma[\alpha(m_o)r]=\gamma[\alpha(m_or)]=\gamma[\alpha(k)]=\gamma(0)=0$. Thus $k\in ker(\beta)$ and the claim is proved. Hence (4) gives $\beta=\varphi\alpha$ for some $\varphi\in S$. Therefore $\varphi(m)=\varphi[\alpha(m_o)]=\beta(m_o)=\gamma(m)$. This shows that φ extends γ.



If M_R is R, the last proposition takes the following form.

Corollary 1.1.12 [14] : The following conditions are equivalent for a ring R .

(1) R is P-injective as R-module.

(2) $ann_R\ ann_R\ (a)= Ra$ for all a in R

(3) $ann_R\ (b)\subseteq ann_R\ (a)$ where a, b in R, implies $Ra\subseteq Rb$.

(4) $ann_R\ [bR\cap ann_R\ (a)\ ]= ann_R\ (b)+Ra$ for all a,b in R

Proof. 3⇒4): Let $x\in ann_R\ [bR\cap ann_R\ (a)\ ]$.

Claim. $ann_R\ (ab) \subseteq ann_R(xb)$, in fact, let $r\in ann_R\ (ab)$, i.e., $(ab)r=0=a(br)$. Therefore $br\in ann_R\ (a) \cap bR$. Since $x\in ann_R[bR\cap ann(a)]$, then $xbr=0=(xb)r$. Hence $r\in ann_R(xb)$. So by (3) $xb\in Rab$, implies $xb=rab$ where $r\in$R.Thus $xb\text{-}rab=0=(x\text{-}ra)b$. Hence $x\text{-}ra\in ann_R(b)$ and so $x\in ann_R(b)+Ra$.

Proposition 1.1.13 [15] : Let M_R be a P.Q.-injective module with S=end (M_R) and let m, n $\in$ M .

(1) If nR is an image of mR, then Sn embeds in Sm

(2) If mR embeds in nR, then Sm is an image of Sn

(3) If mR≅nR, then Sn≅Sm.

Proof : Assume that λ:mR→nR is any R-homomorphism, write λ(m)=na where a$\in$R and define φ:Sn→M by $\varphi[\alpha(n)]=(\alpha n)a=\alpha[\lambda(m)]$ for all $\alpha\in$S. If $\bar{\lambda}\in$S extends λ, then $\varphi[\alpha(n)]=\alpha[\lambda(m)]= \alpha[\ \bar{\lambda}(i(m))]=\alpha[\ \bar{\lambda}(m)]\in$Sm, so φ:Sn→Sm is S-homomorphism. Now to prove (1), if λ is epimorphism, then n=λ(mb) such that b$\in$R.

Given $\alpha(n)\in\ker\varphi$, thus $\alpha(n)=\alpha[\lambda(mb)]=[\alpha\lambda(m)]b$ which implies that $\varphi[\alpha(n)]b$=o.b=o. Hence Sn embeds in Sm. To prove (2), if λ is monomorphism, then $ann_R(\lambda m)\subseteq ann_R(m)$, in fact, let $r\in ann_R(\lambda m)$, then

$\lambda(m)r=\lambda(mr)=0$, so $mr\in\ker(\lambda)$, but λ is monomorphism, then $mr=0$, hence $r\in ann_R(m)$. So by theorem (1.1.9(3)) $m\in S\lambda(m)$, but $S\lambda(m) \subseteq$ image φ. Thus $m \in$ image φ . This means Sm is an image of Sn.
(3) Follows immediately from (1) and (2).

As a special case of the last proposition we have

Corollary 1.1.14 [14] : Let R be a P-injective ring and a, b $\in$R

(1) If aR is an image of bR, then Ra embeds in Rb.

(2) If bR embeds in aR, then Rb is an image of Ra.

Now we need the following definitions.

Definition 1.1.15 [10,p.106] : Let A be a submodule of an R-module M, it is said that M is essential extension of A or (A is an essential submodule of M, i.e., $A\subseteq^{ess} M$ or $A \leq_e M$) or (A is larger in M) if for every non-zero submodule U of M, $A\cap U\neq 0$.

Example 1.1.16: Z_6 as a Z-module. If $A=\{\bar{o}, \bar{2}, \bar{4}\}$, then $A \not\leq_e Z_6$. But if $A=\{\bar{o}, \bar{2}\}\leq Z_4$, then $A \leq_e Z_4$.

Definition 1.1.17 [10,p.212]: Let M be an R-module the sum of all minimal (simple) submodules of M is called the socle of M, equivalently, the intersection of all essential submodules of M, it is denoted by Soc(M). If M has no simple submodule then we put Soc(M)=0. If Soc(M)=M, then M is called semi-simple module.

Example 1.1.18: Z_6 as a Z-module.

$Soc(Z_6)=\{\bar{o}, \bar{2}, \bar{4}\}+\{\bar{o}, \bar{3}\}$. Since $\{\bar{o}, \bar{2}, \bar{4}\}$, $\{\bar{o}, \bar{3}\}$ have no proper submodule except $\{\bar{o}\}$, $\{\bar{o}, \bar{2}, \bar{4}\}$, and $\{\bar{o}\}$, $\{\bar{o}, \bar{3}\}$, respectively, then $Soc(Z_6)=Z_6$. But Soc $(Z_4)=\{\bar{o}, \bar{2}\}$, hence $\{\bar{o}, \bar{2}\}$ the only proper submodule of Z_4. Therefore Z_4 as a Z-module is not semi-simple.

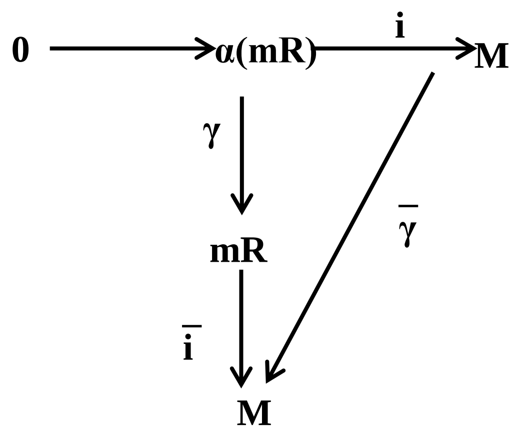
The following result relates $Soc(M_R)$ to $Soc(S^M)$.

Proposition 1.1.19 [15]: Let M_R be a P.Q.-injective module with $S=end(M_R)$.

(1) If mR is a simple R-module, m∈M, then Sm is a simple S-module.

(2) $Soc(M_R) \subseteq^{ess} Soc(S^M)$.

Proof: (1) Consider the following diagram,



We may assume α≠0. Since mR is simple, then α:mR→α(mR) is an isomorphism, let γ: α(mR) →mR be the invers of α, $\bar{i}$, i are inclusion maps from mR, α(mR) to M respectively. Since M is P.Q.-injective module,then there exists $\bar{\gamma}\in S$ that extends γ.Now $\bar{\gamma}[\alpha(m)]= \bar{\gamma}[i(\alpha(m)= \bar{i}[\gamma(\alpha(m))]=\gamma[\alpha(m)]=(\gamma o\alpha)(m)=m$.Hencem∈Sα(m).

(2) This follows from (1).

Proposition 1.1.20 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$, and let $m_1,m_2,\dots,m_n$ be elements of M.

(1) If $Sm_1\oplus\dots\oplus Sm_n$ is a direct sum, then any R-homomorphism $\alpha:m_1R\oplus\dots\oplus m_nR\to M$ has an extension in S.

(2) If $m_1R\oplus\dots\oplus m_nR$ is a direct sum, then $S(m_1+\dots+m_n)= Sm_1+\dots+Sm_n$.

Proof: (1) Let α_i and β denote the restriction of α to $m_i R$and $(m_1+\dots+m_n)R$ respectively and let $\bar{\alpha}_i$ and $\bar{\beta}$ extend α_i and β to M. Then $\Sigma_i\ \bar{\beta}(m_i)= \bar{\beta}(\Sigma_i m_i)=\alpha(\Sigma_i m_i)=\Sigma_i\alpha(m_i)=\Sigma_i\ \bar{\alpha}(m_i)$. Since $\oplus Sm_i$ is a direct , we obtain $\bar{\beta}(m_i)= \bar{\alpha}(m_i)$, in fact, $\bar{\beta}(m_1)+\dots+\bar{\beta}(m_n)= \bar{\alpha}(m_1)+\dots+\bar{\alpha}(m_n)$, so $\bar{\beta}(m_1)-\bar{\alpha}(m_1)= \bar{\alpha}(m_2)+\dots+\bar{\alpha}(m_n)-\bar{\beta}(m_2)-\dots-\bar{\beta}(m_n)\in S_{m1}\cap_{j\neq1}\oplus S_{mj}=0$, then $\bar{\beta}(m_1)-\bar{\alpha}(m_1)=0$ **[10,p.30]**, hence $\bar{\beta}(m_1)= \bar{\alpha}(m_1)$. By the same way we get $\bar{\beta}(m_i)= \bar{\alpha}(m_i)=\alpha(m_i)$, so $\bar{\beta}$ extends α .

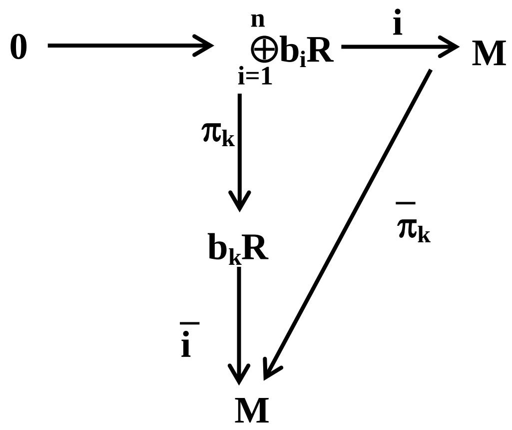
(2)Define α_i: $(m_1+\dots+m_n)R\to M$ by $\alpha_i[(m_1+\dots+m_n)r]=m_i r\ \forall\ r\in R$. Then α_i is well defined. Since M is P.Q.-injective module, then there exists $\bar{\alpha}_i\in S$ that extends α_i , hence $m_i = \alpha_i\ (\Sigma_i m_i)= \bar{\alpha}_i\ [i(\Sigma_i\ m_i)]= \bar{\alpha}(\Sigma_i m_i)\in S(\Sigma_i m_i)$ and it follows that $\Sigma_i Sm_i\subseteq S(\Sigma_i m_i)$. The reverse inclusion always holds.

To prove the next result we need the following defintion **[21]**. A submodule N of an R-module M is said to be fully invariant if for each endomorphism $f:M\to M$, $f(N)\subseteq N$.

For example every submodule of Z as a Z-module is fully invariant. But Z as a submodule of Q is not fully invariant. More over, it is known that every submodule of a multiplication R-module is fully invariant **[17]**.

Proposition 1.1.21 [15] : Let M_R be a P.Q.-injective module with S=end (M_R), and let A, B_1, B_2, …,Bn be fully invariant submodules of M_R. If $B_1\oplus\ldots\oplus B_n$ is a direct, then $A\cap(B_1\oplus\ldots\oplus B_n)=(A\cap B_1)\oplus\ldots\oplus(A\cap B_n)$.

Proof : It is known and is easy to check that $\oplus_i (A\cap B_i)\subseteq A\cap(\oplus_i B_i)$. Now consider the following diagram.

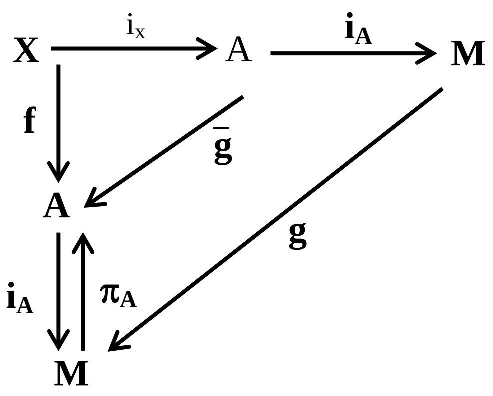


Let $a=\Sigma_I b_i \in A\cap[\oplus_i B_i]$ and let $\pi_k : \bigoplus_{i=1}^{n} b_iR\to b_kR$ be the projection map and i, $\bar{i}$ are inclusion maps from $\bigoplus_{i=1}^{n} b_iR$ and b_kR to M respectively. Sinec $\oplus Sb_i$ is a direct sum, then by prposition (1.1.20) each π_k has an extension $\bar{\pi}_k$ in S, i.e., $\bar{\pi}_k [i(a)]= \bar{i}[\pi_k(a)]$. Since A is fully invariant, then $\bar{\pi}_k (a)= \bar{\pi}_k [i(a)]= \bar{i}[\pi_k (a)]= \pi_k (a)=b_k\in A\cap B_k$ for each k whence $a \in\oplus_i(A\cap B_i)$.

Proposition 1.1.22 [15] : Every summand of a P.Q.-injective module is a gain P.Q.-injective module

Proof : Let $M=A\oplus B$ be a P.Q.-injective module and let X be a principal submodule of A, with f a homomorphism of X into A, let i_x and

i_A be the inclusion maps of X in A and A in M respectively and π_A: M→A be the projection map .Consider the following diagram.



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Since M is P.Q8.-injective module , then there exists a homomorphism g:M→M such that $g o i_A o i_x = i_A o f$.

Define $\bar{g} = \pi_{A o} g_o i_A$, then $\bar{g}$ is a homomorphism of A into A. Note that $\bar{g}$ extends f, that is $(\bar{g}_o i_x)_{(x)} = \bar{g}[i_x\,(x)] = \bar{g}(x) = g(x) = (i_{Ao} f)_{(x)} = f(x)$.

2

Section 1.2 The Jacobson Radical and Related Concepts

Recall that the intersection of all maximal submodules of M_R is called the Jacobson radical of M it is denoted by J(M). If M has no maximal submodule, then we put J(M)=**M[10,p.213]**.

For S=end(M_R), we define J(S) to be the Jacobson radical of the 8S2-module. Recall that an R-module M is called singular module if Z(M)=0 where Z(M)={$x\in M\backslash ann(x) \leq_e R$} and non-singular if Z(M)=M **[9]**. More over recall that W(S)={$w\in S\backslash ker(w)\leq_e M$} where S=end($M_R$) **[15]**. In this section we study the relation between J(M), Z(M) and W(S).

Examples 1.2.1 :

(1) J(Z_4)={ $\bar{0}$, $\bar{2}$}, but J(Z_6)=0.

(2) Q as a Z-module is singluar module, i.e., ifo$\neq x\in$Q, ann(x)$\not\leq_e$ Z, then Z(Q)=0, on the other hand Z_n as a Z-module is non-singular module, i.e., Z(Z_n)=Z_n.

Lemma 1.2.2 [12,p.38]: Let M be an R-module with S=end(M_R). Then W(S)={$w\in S\backslash$ ker(w)$\leq_e$M} is a two sided ideal in S.

Proof: Let a ,b $\in$W(S) and $\alpha\in$S. Then kera $\leq_e$ M and kerb $\leq_e$ M. Since ker a$\cap$ker b $\leq$ ker (a-b) and ker a $\leq$ ker αa, ker (a-b) and ker αa are essential submodules of M and consequently , a-b $\in$W(S) and αa$\in$W(S). Let N={$n\in M\backslash\alpha(n)\in$ ker a}. Then it is clear that N$\leq_e$ M and N$\leq$ ker aα. Hence a$\alpha\in$W(S).

Remark 1.2.3 : If w$\in$W(S), then ker (w) $\cap$ ker (1-βw)=0, for all $\beta\in$S.

Proof : Let $x \in \ker(w)$ and $x \in \ker(1-\beta w)$ then $w(x)=0$ and $(1-\beta w)_{(x)}=0$, hence $x=0$.

Remark 1.2.4 [15] : $W(S) \subseteq \{w \in S \backslash 1-\beta w$ is monomorphism for all $\beta \in S\}$.

Proof : Since $w \in W(S)$, then $\ker(w) \cap \ker(1-\beta w)=0$ and $\ker w \leq_e M$, so by definition (1.1.15).Ker $(1-\beta w)=0$. Therefore $1-\beta w$ is monomorphism. Thus $W(S) \subseteq \{w \in S \backslash 1-\beta w$ is monomorphism for all $\beta \in S\}$.

The following proposition shows that equality in (1.2.4) holds for P.Q.-injective module.

Proposition 1.2.5 [15] : If M_R is a P.Q.-injective module then $W(S)=\{w \in S \backslash 1-\beta w$ is monomorphism for all $\beta \in S\}$.

Proof: Assume that $1-\beta w$ is monomorphism for all $\beta \in S$, and let $\ker(w) \cap mR=0$, $m \in M$. then $ann_R(wm) \subseteq ann_R(m)$, in fact, let $r \in ann_R(wm)$, then $w(mr)=w(m)r=0$. Hence $mr \in \ker(w) \cap mR=0$. Thus $mr=0$, so $r \in ann_R(m)$. By theorem (1.1.9(3)) $m \in Swm$. i.e., $m=\beta(wm)=m-\beta wm=(1-\beta w)(m)=0$. This means that $m \in \ker(1-\beta w)$ for some $\beta \in S$, but $(1-\beta w)$ is monomorphism, so $m=0$. This proves that $w \in W(S)$. The other inclusion follows from the last remark.

Before the next lemma we need this definition.

Definition 1.2.6 [15] : The module M_R is called a kasch module if every simple subquotient of M embeds in M.

For example, let $M=Z_6=Z_2\oplus Z_3$. Since $Z_6/Z_2\cong Z_3$, then Z_3 embeds in Z_6, i.e., there exists a monomorphism $f:Z_3\rightarrow M$, similary for Z_6/Z_3. But Z as a Z-module is not kasch module.

For example $Z/2Z\cong Z_2$ does not embeded in Z.

We need the following lemma to prove the next theorem.

Lemma 1.2.7 [15] : Let M_R be a P.Q.-injective module which is a kasch module. If T is maximal ideal of R (it is denoted by $T\subset^{max} R$), then $ann_M(T)\neq 0$ if and only if $ann_R(m)\subseteq T$ for some $0\neq m\in M$. In this case $ann_M(T)$ is a simple left S-module.

Proof : If $0\neq m\in ann_M(T)$, then $mT=0$, hence $T\subseteq ann_R(m)\neq R$, so $T=ann_R(m)$ by the maximality of T, which implies $ann_R(m)\subseteq T$.Conversely, assume $ann_R(m)\subseteq T$ where $0\neq m\in M$, observe first that $mR\neq mT$, in fact, if $m.l=mt$ where $t\in T$, then $m(1-t)=0$, hence $1-t\in ann_R(m)\subseteq T$, implies that $1-t+t\in T$, so $l\in T$ contradiction with maximality of T. Hence choose $\frac{x}{mT}\subset^{max}\frac{mR}{mT}$. As M is kasch. Let $6:\ \frac{mR}{x}\rightarrow M_R$ be a monomorphism and write $m_o=6(m+x)$. Then $0\neq m_o\in ann_M(T)$, implies that $m_oT=6(m+x)T=6(mT+x)=6(0+x)=6(x)=0$. Finally, let $0\neq m_1\in ann_M(T)$, then $m_1T=0$, hence $T\subseteq ann_R(m_1)$ whence $T=ann_R(m_1)$. Thus $ann_M(T)= ann_M ann_R(m_1)=Sm_1$by theorem (1.1.9(2)). Hence $ann_M(T)$ is simple as a left S-module, that proves the lemma

Theorem 1.2.8 [15] : Let M_R be a P.Q.-injective module which is a kasch module with $S=end(M_R)$. Then

(1) $Soc(M_R)=Soc(S^M)\subseteq ann_M(J(S))$.

(2) $Soc(S^M) \subseteq^{ess} S^M$.

Proof : By proposition (1.1.19(2) we have $Soc(M_R) \subseteq Soc(S^M)$ we show $Soc(S^M) \subseteq ann_M(J)$, in fact, let s^N be a simple submodule of the S-module M. Since every simple submodule is cyclic, then Ns is cyclic, in particular, Ns is finitely generated. Since every finitely generated module has maximal submodule **[10,p.28]** , but Ns is simple, then either JNs=Ns contradiction with maximality or JNs={0}, then JNs=0. Hence $Ns \in ann_M(J)$. Now let $0 \neq m \in M$, if $ann_R(m) \subseteq T \subset^{max} R$, then $ann_M(T) \subseteq ann_M ann_R(m)$, in fact, let $x \in ann_M(T)$, then xT=0 where $x \in M$, we want to show that $x \in ann_M ann_R(m)$, i.e., xr=0 where rm=0. Since $T \subseteq ann_R(m)$, then tm=0. Hence xr=0, implies that $x \in ann_M ann_R(m)$. Thus by theorem (1.1.9(2)) $ann_M(T) \subseteq ann_M ann_R(m) = Sm$. As $ann_M(T)$ is simple by lemma (1.2.7), this shows that $Soc(S^M) \subseteq^{ess} S^M$. This proves (2). Finally to show that $Soc(S^M) \subseteq Soc(M)$, let Sm be a simple module and let $ann_R(m) \subseteq T \subset^{max} R$. Since $ann_M(T) \neq 0$ by lemma (1.2.7), then $ann_M(T) \subseteq ann_M ann_R(m) = Sm$, but Sm is simple, then $Sm = ann_M(T)$. Thus $T \subseteq ann_R ann_M(T) = ann_R(Sm) = ann_R(m) \neq R$, it is clear that $R/ann_R(m) \cong mR$. Since T is maximal, $ann_R(m)=T$ whence mR=R/T is simple. It follows that $Soc(S^M) \subseteq Soc(M_R)$.

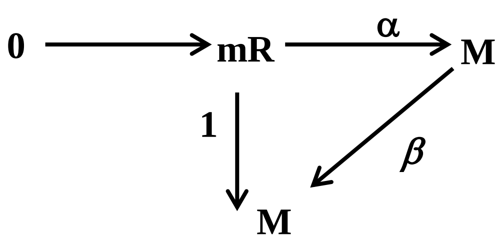
Proposition 1.2.9 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$. Then

(1) $Z(S_S) \subseteq W(S)$ and $J(S) \subseteq W(S)$.

(2) If every monomorphism in S has a left inverse then $W(S) \subseteq J(S)$.

Proof :

(1) Suppose $\alpha \in W(S)-Z(Ss)$, then $\ker(\alpha) \underset{e}{\trianglelefteq} M_R$, thus $\ker(\alpha)\cap mR=0$ where $0\neq m\in M$. Hence $\alpha|_{mR}:mR\rightarrow M$ is monomorphism, then by theorem (1.1.9(4)) there exists $\beta:M\rightarrow M$ such that $(\beta\circ\alpha)=1_{mR}$ which implies $(1-\beta\alpha)_{(m)}=0$. Thus $m\in\ker(1-\beta\alpha)$, hence $\ker(1-\beta\alpha)\neq 0$ contradicting proposition (1.2.5). Hence $Z(Ss)\subseteq W(S)$



(2)$\forall$ $w\in W(S)$, $\ker(w)\cap\ker(1-\beta w)=0$ for all $\beta\in S$ by remark (1.2.3), thus if $\ker(1-\beta w)=0$, then $1-\beta w$ is monomorphism. Thus by hypothesis, $1-\beta w$ has a left inverse, so by **[10, p.220]** , $w\in J(S)$. Hence $W(S)\subseteq J(S)$.

Theorem 1.2.10 [14] : If R is P-injective ring, then $J(R)=Z(R_R)$

Proof : If $a \in Z(R)$, then $ann(a)\underset{e}{\leq} R$. More over, it is easily seen that $ann(1-a)=0$. Hence by corollary (1.1.12(2)) $R=ann\ ann(1-a)=R(1-a)$ and thus $R=R(1-a)$ which shows that $Z(R_R)\subseteq J(R)$. Conversely, if $a\in J(R)$ we show that $bR\cap ann(a)=0$ where $b\in R$, implies that $b=0$. But by corollary (1.1.12(4)) $ann_R(b)+Ra=ann_R[bR\cap ann(a)]=R$, so $ann(b)=R$.

Proposition 1.2.11 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$. If M is non-singular, then $w(S)=J(S)=0$.

Proof : By proposition (1.2.9) $J(S)\subseteq W(S)$, thus it is enough to show that $W(S)=0$. If $w\in W(S)$, then $\ker(w)\underset{e}{\leq} M_R$.Since M is non-singular, then by **[9]** M has no proper essential submodule, which implies that $\ker(w)=M_R$ and $w=0$.

<u>Definition 1.2.12 [15]</u> : A module M_R is said to satisfy the C_2-condition, if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M, i.e., $N\leq M$, $N\cong K$ where $M=K\oplus J$ and $J\leq M$, then $M=N\oplus L$ where $L\leq M$.

Recall that an R-homomorphism $f:A\to B$ (where A and B are two R-modules) is said to split if there exists an R-homomorphism $g:B\to A$ such that $g\circ f=I_A$ **[10, p.115]** .

Now we show that there exists a relation between the C_2-condition and cyclic P.Q.-injective modules.

<u>Proposition 1.2.13 [15]</u> : Let M_R be a P.Q.-injective module with $S=\text{end}(M_R)$.

(1) If N and K are isomorphic cyclic submodules of M and K is a direct summand, then N is also a direct summand.

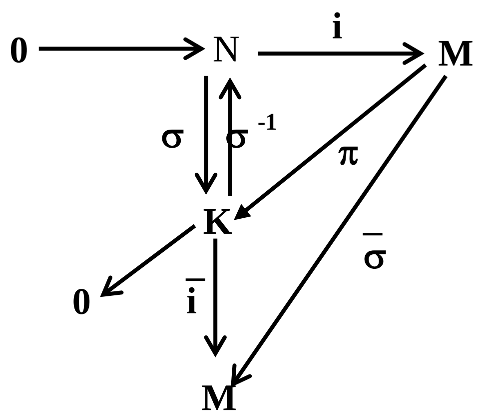
(2) Every cyclic P.Q.-injective module satisfies the C_2-condition.

<u>Proof :</u> Since a direct summand of a cyclic module is cyclic, it is enough to prove (1), now let $\sigma:N\to K$ be an isomorphism and $\pi:M\to K$ be the projection. If $\bar{\sigma}:M\to M$ is an extension of σ, put $\alpha=\sigma^{-1}\circ\pi\circ\bar{\sigma}:M\to N$.

Thus $\sigma(n)=k\in K$, so $\alpha(n)=\sigma^{-1}[\pi(\bar{\sigma}(n)]=\sigma^{-1}[\pi(\bar{\sigma}(i(n)))]=$ $\sigma^{-1}[\pi(\sigma(\bar{i}(n)))]=\sigma^{-1}[\pi(\sigma(n))]=\sigma^{-1}[\pi(k)]=\sigma^{-1}(k)=\sigma^{-1}[\sigma(n)]=(\sigma^{-1}\circ\sigma)_{(n)}=n$. Hence the inclusion map $N\to M$ splits, i.e., $\alpha\circ i=1_N$

$$0 \longrightarrow N \underset{\alpha}{\overset{i}{\rightleftarrows}} M \xrightarrow{\pi} K \longrightarrow 0$$

This means, the sequence is split. Hence by **[10, p.116]** N is isomorphic to a direct summand of M. This proves (1).



The following lemma appeared in **[19, 41.22]**.

Lemma 1.2.14 : Let M_R be any module with S=end(M_R) . If M has C_2-condition, then W(S)⊆J(S).

The following proposition is known for quasi-injective modules.

Proposition 1.2.15 [15] : If M_R is a cyclic P.Q.-injective module, then J(S)=W(S).

Proof : Since M is P.Q.-injective module, then by proposition (1.2.9(1)) J(S)⊆W(S). Since M is cyclic P.Q.-injective module , then by proposition (1.2.13(2)) and lemma (1.2.14) W(S)⊆J(S). Hence W(S)=J(S).

Proposition 1.2.16 [15] : Let M_R be a principally self-generator with S=end(M_R) . Then Z(Ss)=W(S).

Proof : Let w∈W(S). Given $0\neq\beta\in S$ we have ker (w)$\cap\beta$(M)≠0, thus there exists $0\neq\beta(m_o)\in$ker(w), i.e., w[$\beta(m_o)$]=0 . Since M is principally self-generator, then $m_o=\lambda(m_1)$ where $\lambda:M\to m_oR$, so $\beta(m_o)=\beta[\lambda(m_1)]$. This means $\beta\,\lambda\neq0$, but $w\beta\lambda=0$ because $w\beta\lambda(M)\subseteq w\beta(m_oR)=w\beta(m_o)R$

$=0$, hence $0\neq\beta\lambda\in ann_S(w)\cap\beta S$, proving that $w\in Z(Ss)$. Conversely, if $w\in Z(Ss)$ and $0\neq m_o\in M$, we must show that $\ker(w)\cap m_oR\neq0$. Since M is principally self-generator, then there exists $\lambda:M\to m_oR\to0$. Then $\lambda\neq0$, so $ann_S(w)\cap\lambda S\neq0$.

Put $w\lambda\beta=0$, i.e., $\lambda\beta\in ann_S(w)\cap\lambda S$ for some $\beta\in S$ where $\lambda\beta\neq0$, let $\lambda\beta(m_1)\neq0$ where $m_1\in M$, we have $\lambda\beta(m_1)\in\lambda(M)=m_oR$, so write $\lambda\beta(m_1)=m_oa\ \forall\ a\in R$. Then $w(m_oa)=w[\lambda\beta(m_1)]=0$, so $w(m_oa)=0$, hence $0\neq m_oa\in\ker(w)\cap m_oR$. This shows that $w\in W(S)$.

Proposition 1.2.17 : Let M_R be a P.Q.-injective module. If M is cyclic, then $Z(Ss)\subseteq J(Ss)$.

Proof : Since M is P.Q.-injective module, then by proposition (1.2.9(1)) $Z(Ss)\subseteq W(S)$. More over by proposition(1.2.13(2)) M has C_2-condition, so by lemma (1.2.14) $W(S)\subseteq J(S)$. Hence $Z(Ss)\subseteq J(Ss)$.

Section 1.3 Further Results On Principally Injective Rings

In this section we give further results on principally injective rings. For reference see **[14]**.

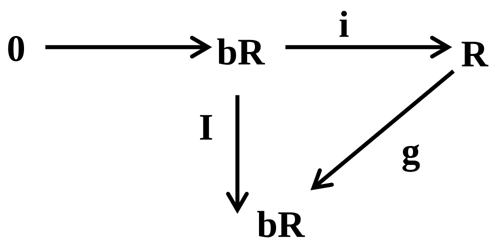
Recall that the ring R is principally injective if it is principally injective as an R-module.

Example 1.3.1 [7] : Let R be the ring generated over the field Z_2 by variables x_1, x_2, …where $x_i^3 = 0$ and $x_i^2 = x_j^2$ for all i,j . Then R is commutative, principally injective , but not injective.

A statemenet similar to the following statement is known for pointwise injective modules **[3]**, we prove it for P-injective rings.

Proposition 1.3.2 : Let R be a ring in which every cyclic R-module is P-injective, then R is regular ring.

Proof : For any b∈R, consider the following diagram.



Where I:bR→bR is the identity R-homomorphism and i:bR →R the inclusion map. Since bR is P-injective , then there exists g:R→bR such that I(b)=(g_oi)(b), hence b=I(b)=(g_oi) (b)=g(b)=g(1)b. Since g(1)∈bR, then g(1)=ba for some a∈R. which shows that b=bab. Therefore R is regular ring.

The next result shows that the C_2-condition and C_3-condition **[14]** hold in a P-injective rings. Where a module M is said to satisfy the

C_3-condition, if whenever N and K are direct summands with $N \cap K=0$, then N+K is also a direct summand **[15]** .

Theorem 1.3.3 [14] : Let R be a P-injective ring and let a, $b \in R$.

(1) If $aR \cong bR$ and bR is a direct summand of R, then aR is a direct summand of R.

(2) If each of aR and bR is a direct summand of R and $aR \cap bR=0$, then $(aR \oplus bR)$ is a direct summand of R.

Proof :

(1) This follows from proposition (1.2.13(1)).

(2) Since aR is a direct summand of R, then aR=eR with $e^2=e$, so that $aR \oplus bR=eR \oplus (1-e)bR$. Hence $(1-e)bR \cong bR$, so by (1) (1-e)bR=gR where $g^2=g$. Since R is commutative, then eg=0, so h=e+g-ge is an idempotent element, in fact, $h^2=(e + g - ge)^2=(e + g - ge)\ (e + g - ge)=e^2 + eg - ge^2 + ge + g^2 - g^2e - ge^2 - g^2e + g^2e^2=e + eg - ge + ge + g – ge – ge + ge + ge = e + g - ge$. Hence $aR \oplus bR=hR$.

Before the next result we give some definitions.

Definition 1.3.4 [10,p.124] : Let M be an R-module, a monomorphism $\sigma:M \rightarrow E$ is called an injective hull of M if E is injective and σ is essential monomorphism, i.e., $\sigma(M) \leq_e E$.

For example, Q_Z is an injective hull of Z_Z.

Remark 1.3.5 : Let M be an R-module, then every module has an injective hull **[10, p.127]**

Note 1.3.6 : We use the notation I(M) for injective hull of M.

Definition 1.3.7 [14] : An R-module M is called weakly injective if for every finitely generated submodule N⊆I(M), N⊆X⊆I(M) for some X≅M.

Remark 1.3.8 : Every injective module is weakly injective module, but the converse is not true .

For example, the Z-module Z_2 is not weakly injective, in fact ,$I(Z_2)= Z_{2^\infty}$ and $Z_4 \subseteq Z_{2^\infty}$, but $Z_4 \ncong Z_2$. However, the Z-module Z is weakly injective but not injective. In fact, I(Z)=Q, and every finitely generated Z-submodule of Q has the form $\frac{Z}{b}=\{\frac{n}{b} \setminus n \in Z, b \neq 0\}$, clearly $\frac{Z}{b} \cong Z$ and $\frac{Z}{b} \subseteq \frac{Z}{b} \subseteq Q$.

Theorem 1.3.9 [14] : R is self-injective if and only if R is P-injective and weakly injective.

Proof : The conditions are clearly necessary. For the converse, if a∈I(R) we show that a∈R. we have R+aR⊆X⊆I(R) with X≅R. Hence X has the C_2 –condition (property(1) Theorem (1.3.3), so R is a direct summand of X. But R is essential in I(R), so R=X as required.

Definition 1.3.10 [10,p.52] :

(1) An R-module B is called a generator of an R-module M if M=Σ Im(φ) where φ∈Hom (B,M) .

(2) An R-module C is called a cogenerator of an R-module M if 0=∩kerφ where φ∈Hom(M,C).

Recall that an R-module M is called a duo module if every submodule of M is fully invariant **[21]** .

Theorem 1.3.11 [14] : Let M be a duo R-module with $S=end(M_R)$, let β, γ denote elements of S.

(1) Assume that M generator $\ker\beta$ for each $\beta\in S$. Then S is P-injective if and only if $\ker\beta\subseteq\ker\gamma$ implies that $\gamma\in S\beta$.

(2) Assume that M cogenerates $M/\beta M$ for each $\beta\in S$. Then S is P-injective if and only if $\gamma M\subseteq\beta M$, implies that $\gamma\in\beta S$.

Proof (1) ⇒) Since M is duo module, then it is easily seen that S is commutative. S is P-injective, then by proposition (1.1.11) if $\ker\beta\subseteq\ker\gamma$, then $\gamma\in\beta S$, this condition holds for any M.

⇐) if $\gamma\in ann_S ann_S(\beta)$, i.e., $\gamma[ann_S(\beta)]=0$, we show that $\ker\beta\subseteq\ker\gamma$. Let $x\in\ker\beta$, since M generates $\ker\beta$, then $x=\Sigma_i\ \lambda_i(m_i)$ where $\lambda_i:M\to\ker\beta$, hence $\beta(x)=\beta(\lambda_i)=0$ for each i , which implies $\lambda_i\in ann_s(\beta)$. Thus $\gamma\lambda_i=0$, it follows that $x\in\ker\gamma$. Hence by proposition (1.1.11) S is P-injective.

(2)⇒) Again, the forward implication always holds.

⇐) If $\gamma\in ann_S ann_S(\beta)$, then $\gamma\alpha=0$, where $\alpha\in ann_S(\beta)$, so $\alpha\beta=0$, hence $\gamma\beta=0$, we want to show that $\gamma M\subseteq\beta M$, assume not, then there exists $m_o\in M$ such that $\gamma(m_o)\notin\beta M$. Since M cogenerates $M/\beta M$, then there exists $\sigma:M/\beta M\to M$ satisfies $\sigma[\gamma(m_o)+\beta(M)]\neq 0$. If $\lambda:M\to M$ is defined by $\lambda(m)=\sigma[m+\beta(M)]$, then $\lambda[\gamma(m_o)]=\sigma[\gamma(m_o)+\beta(M)]\neq 0$, hence $\lambda\gamma\neq 0$, so $\lambda\ [\beta\ (m)]=\ \sigma[\beta(m)+\beta(M)]$, but $\beta(m)\in\beta(M)$. Thus $\lambda\beta(m)=\sigma[\beta(M)]=0$, therefore $\lambda\beta=0$ a contradiction. Hence $\gamma M\subseteq\beta M$, which implies that $\gamma\in\beta S$. Then by proposition (1.1.11)S is P-injective.

Befor giving the next lemma we give this definition.

Definition 1.3.12 [10,p.147] : An asending chain of submodules of the form $N_1\subseteq N_2\subseteq\ldots\subseteq N_n\subseteq\ldots$ is said to satisfy the asending chain condition if there exists $n\in N$ such that $N_n= N_{n+1}=\ldots$.

Remark 1.3.13 : We use the notation A.C.C. for asending chain condition.

We need the following lemma .

Lemma 1.3.14 [14] : Let R be a ring and let I be an ideal of R such that R/I satisfies the A.C.C. on annihilators. If $y_1, y_2, \ldots$ are subsets of ann (I), then there exists n ≥1 such that ann $(y_{n+1} \ldots y_1)$=ann $(y_n \ldots y_1)$ where y_i y_j is the set theortic product of y_i y_j.

Proof: Write $\overline{R}$=R/I and $r \to \overline{r}$ denote the natural homomorphism $R \to \overline{R}$. Then ann $(\overline{y}_1) \subseteq$ann$(\overline{y}_2\ \overline{y}_1) \subseteq$ann$(\overline{y}_3\ \overline{y}_2\ \overline{y}_1) \subseteq \ldots$. Since $\overline{R}$=R/I satisfies A.C.C. on annihiliators, then ann $(\overline{y}_{n+1}\ \overline{y}_n \ldots \overline{y}_1)$ =ann$(\overline{y}_n \ldots \overline{y}_1)$ for some n≥1 . Now if a∈ann$(y_{n+1} \ldots y_1)$, then $(y_{n+1} \ldots \ldots y_1)$ a=0. Since $\overline{R}$=R/I and y_1 y_2 …are subsets of ann (I), then $\overline{R}$=$(0+y_{n+1}) \ldots . (0+y_1)$ (0+a)= $\overline{y}_{n+1}\ \overline{y}_n \ldots \overline{y}_1\ \overline{a} = \overline{0}$, so $\overline{y}_n \ldots \overline{y}_1\ \overline{a}$ =0 and $y_n \ldots y_1 a \subseteq$ I. Since every A.C.C. has maximal element **[10, p.147]** , then I $\subseteq$ ann (y_{n+1}). Thus $y_n \ldots y_1 a \subseteq$ann$(y_{n+1})$, i.e., y_{n+1} $(y_n \ldots y_1 a)$=0 proving this lemma.

Next we need the definition of T-nilpotent ideal.

Definition 1.3.15 [10, p.291] : A set A of a ring R is called T-nilpotent if for every family $(a_1, a_2, \ldots .)$, $a_i \in A$ a $k \in N$ exists with a_k $a_{k-1} \ldots . a_1 = 0$, a_1 $a_2 \ldots . a_k = 0$.

The following result was proved by Armendariz and Park **[2]** .

Theorem 1.3.16 [14] : If R is P-injective and R/Soc(R) satisfies the A.C.C. on annihilators, then J(R) is nilpotent .

Proof : Assume J=J(R) and K=Soc (R).Since R is P-injective, then by n theorem (1.2.10) JK=0, so J$\subseteq$ann (K). It suffices to show that J is

T-nilpotent . (J+K)/K is nilpotent in R/K by hypothesis. Now let $a_1,a_2, \ldots$ be given in J, we show that $a_n \ldots a_2 a_1=0$ for some n . Sinece R/Soc(R) satisfies the A.C.C., then by lemma (1.3.14) $ann(a_{n+1} a_n \ldots a_1)=ann(a_n \ldots a_1)$ for some n . So by corollary (1.1.12) $Ra_{n+1} a_n \ldots a_1=Ra_n \ldots a_1$. Hence $ra_{n+1} a_n \ldots a_1=a_n \ldots a_1$ where $r \in R$. So $a_n \ldots a_1-ra_{n+1} a_n \ldots a_1=0$, which implies $a_n \ldots a_1(1-ra_{n+1})=0$. Since $ra_{n+1} \in J(R)$ if and only if $1-ra_{n+1}$ is invertible **[10,p.220]**, then there exists $t \in R$ such that $a_n \ldots a_1(1-ra_{n+1})t=0$, so $(1-ra_{n+1})t=1$, then $a_n \ldots a_1=0$.

Corollary 1.3.17 [14] : If R is P-injective and satisfies the A.C.C. on annihilators, then J(R) is nilpotent .

Proof: see [13] .

CHAPTER TWO

PRINCIPALLY QUASI-INJECTIVE MODULES AND OTHER CLASSES OF MODULES

Introduction:

In this chapter, we study the relation between the class of principally quasi-injective modules and other well-known classes of modules.

In section 1, we study the relation between principally quasi-injective modules and summand intersection property, summand sum property. For references **[15],[4],[5]**.

In section 2, we study the relation between duo principally quasi-injective modules and uniform submodules.

In section 3 , we study the relation between principally quasi-injective modules and continuous modules where an R-module M is continuous if M has C_1-condition **[8]** and C_2-condition **[15]** .

Section 2.1 The Endomorphism Ring of a Principally Quasi-Injective Module.

In this section we recall the definition of modules with summand intersection property (SIP) and summand sum property (SSP), and we look at some properties of these modules. For more details see **[15],[4],[5]**. We also study the relation between the module M being uniform and the ring of endomorphisms being local **[15]** .

Recall that an R-module M is called uniform if every submodule of M is essential in M, and the submodule U of M is essential if for every non-zero submodule A of M, $A \cap U \neq 0$.

For example, the Z-module Z_6 is not uniform where Z_4 as a Z-module is uniform .

Recall that a ring R is called local ring if it has one unique maximal ideal.

Remark 2.1.1 [10, p.169] : A ring R is local ring if and only if the set of non-units of R is an ideal in R.

Proposition 2.1.2 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$.

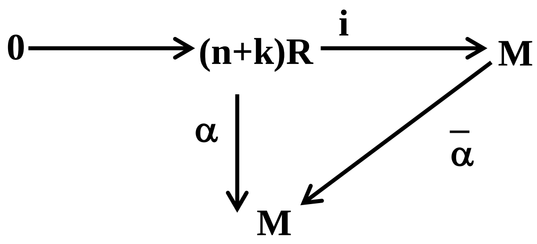
(1) If S is local, then M is uniform.

(2) If M is cyclic and uniform, then S is local.

Proof:

(1) Suppose N and K are non-zero submodules of M such that $N \cap K=0$, choose $0 \neq n \in N$ and $0 \neq k \in K$, define $\alpha:(n+k)R \rightarrow M$ by $\alpha[(n+k)r]=nr$. This is well-defined, in fact, let $(n+k)r_1 = (n+k)r_2$ where $r_1,r_2 \in R$, so $(n+k)r_1-(n+k)r_2=(n+k)r_1-r_2 \in N \cap K=0$, then $(n+k)(r_1-r_2)=0$. Hence $\alpha[(n+k)(r_1-r_2)]= n(r_1-r_2)=0$. Therefore $n(r_1-r_2)=nr_1-nr_2=0$, so $nr_1=nr_2$ .Since M is P.Q.-injective, then there

exists $\bar{\alpha} \in S$ that extends α. Hence $(1-\bar{\alpha})(n)=0=\bar{\alpha}(k)$. Since S is local, then either $\bar{\alpha}$ is a unit or $(1-\bar{\alpha})$ is a unit, i.e., n=0 or k=0 which is a contradiction.



(2) Since M is cyclic, then by proposition (1.2.15) W(S)=J(S). Now if $\alpha \in S$ is a non-unit, then ker $(\alpha) \neq 0$. Since M is uniform, then ker $(\alpha) \leq_e M_R$. Hence $\alpha \in W(S)=J(S)$, so J(S) contains all non-units of S . Thus by remark (2.1.1) S is local.

Definition 2.1.3 [10, p.281] : A ring R is called semiperfect if $\bar{R}=R/rad(R)$ is semisimple and every idempotent element $s \in \bar{R}$ there is an idempotent element $e \in R$ with s= e

Proposition 2.1.4 [15] : If M is a finite direct sum of submodules with local endomorphism rings, then S=end(M) is semiperfect .

The converse holds for P-injective duo modules.

Recall that an R-module M is multiplication R-module where R is commutative if for every submodule N of M, there exists an ideal I of R such that N=IM. It is known that every multiplication module is a duo module **[17]**.

The following proposition is well-known, but we present here a proof for the sake of completeness.

Proposition 2.1.5 [8] : Let M be a duo R-module and A is a direct summand. Then

(1) A is itself a duo module .

(2) If M is a self-generator, then A is also a self-generator .

Proof :

(1) Let $\varphi \in end(A)$, $\pi:M \to A$, $i:A \to M$ be the projection and imbedding respectively. Then $\Psi = i_o \varphi_o \pi \in end(M)$. Since M is duo and $i_o \pi = 1_A$, then for any submodule X of A, $\varphi(x) = \Psi(x) \subset X$, proving that A is duo.

(2) Since A is a direct summand, then $M = A \oplus B$ where $B \leq M$, hence for any $\varphi \in end(M)$ we get $\varphi(M) = \varphi(A+B) = \varphi(A) + \varphi(B)$. Since M is a self-generator, then X can be written as $X = \sum_{\varphi \in I} \varphi(M) = \sum_{\varphi \in I} (\varphi(A) + \varphi(B))$ for some subset I of end (M) where X is a submodule of A . Since M is duo, then $\varphi(B) \subseteq B$, it follows that $\varphi(B) = 0$ for all $\varphi \in I$. Hence $X = \sum_{\varphi \in I} \varphi(A)$. More over, φ can be considered as an endomorphism of A, since $\varphi(A) \subseteq A$. This shows that A is a self-generator.

The following result gives a relation between the module M being duo and the ring $S = end(M_R)$ being semiperfect.

Proposition 2.1.6 [15] : Let M_R be a duo, P.Q.-injective module for which $S = end(M_R)$ is semiperfect. Then M is a finite direct sum of uniform P.Q.-injective modules.

Proof: Since S is semiperfect, then $M = M_1 \oplus \ldots \oplus M_n$ where $S = end(M_R)$ is local for each i . Since M is duo then each M_i is duo and P.Q.-injective. Hence by proposition (2.1.2) M_i is uniform.

Recall that an R-module M is said to have the summand intersection property (SIP) if the intersection of any two direct summands is a gain a direct summand **[15],[4],[5]**.

Examples 2.1.7 :

(1) Every multiplication R- module has the SIP **[5]** .

(2) In particular every commutative ring with identity has the SIP, in fact, assume $R=A\oplus A_1=B\oplus B_1$ where A, B, A_1, $B_1\subseteq R$. Since A and B are summands of R, then A=Re and B=Rf such that e and f are idempotent elements in R, then by **[10, p.174]**, it is easy to check that $A\cap B=Ref$. Hence $A\cap B$ is a direct summand in R .

(3) Consider the module $M=Z_4\oplus Z_2$ as a Z-module, hence M={(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)}, let $A=Z_4 \oplus 0$ and B=Z(1,1), the submodule generated by (1,1). Now A and B are summands of M. But $A\cap B$={(0,0), (2,0)} is not a summand of M. Thus M does not have the SIP .

Proposition 2.1.8 [15] : Let M_R be a P.Q.-injective, duo module, then M_R has the SIP .

Proof : Suppose N and K are direct summands of M, i.e., $M=N\oplus N_1$ and $M=K\oplus K_1$ where N_1,K_1 are submodules of M. We must show that $N\cap K$ is a summand of M. Note that $N=N\cap(K\oplus K_1)$. Since M is duo, then by proposition (1.1.21) $N=N \cap (K\oplus K_1) = (N\cap K) \oplus (N\cap K_1)$. Hence $M=N \oplus N_1= (N\cap K) \oplus (N\cap K_1)\oplus N_1$ and so $N\cap K$ is indeed a direct summand.

Recall that an R-module M is said to have the summand sum property (SSP) if the sum of any two summands of M is again a summand **[4],[5],[15]**.

The following proposition shows that there exists a relation between SIP and SSP under the C_3-condition .

Proposition 2.1.9 [15] : If M_R has the C_3-condition and the SIP, then M has the SSP .

Proof : Suppose N and K are direct summands of M. We must show that N+K is a summand. Since M has the SIP, then $N\cap K$ is a direct summand , i.e., $M=(N\cap K)\oplus X$ where X is a submodule of M. Now we have $K=(N\cap K)\oplus(K\cap X)$. So $N+K=N+ [(N\cap K)\oplus(K\cap X)]=N\oplus(K\cap X)$. So N and $K\cap X$ are both summands, then N+K is a direct summand because M satisfies C_3-condition.

Proposition 2.1.10 [15] : Let M_R be a cyclic , P.Q.-injective module. Then M has both the SIP and the SSP.

Proof : Since M is cyclic, then M is multiplication module and hence is duo, then by proposition (2.1.8) M has the SIP. Since M is cyclic, P.Q.-injective, then by proposition (1.2.13(2)) M has C_2-condition, hence M has C_3-condition **[14]**. Thus by proposition (2.1.9) M has SSP.

Section 2.2 Uniform Submodules

In a duo principally quasi-injective module M there is a relationship between the maximal left ideals of the endomorphism ring and the maximal uniform submodules of M. This is explored in this section. Many of the ideas in this section trace back to Camillo **[7]**.

We need the following lemma for the proof of the theorem.

Lemma 2.2.1 [15]: Let M_R be a P.Q.-injective module. Let N be a non-zero submodule of M and let $N\subseteq^{ess}P\subseteq M$ and $N\subseteq^{ess}Q\subseteq M$. If P is fully invariant in M, then $N\subseteq^{ess}P+Q$.

Proof : Suppose $0\neq p+q\in P+Q$. Since $N\subseteq^{ess} Q$, then if $(p+q)R\cap Q\neq 0$, then $(p+q)R\cap N\neq 0$. $(p+q)ann_R(p)\subseteq (p+q)R\cap qR$, in fact , let $x\in(p+q)ann_R(p)$, then $x=(p+q)r$ where $r\in R$ and $pr=0$, hence $x=pr+qr=0 +qr =qr\in (p+q)R\cap qR$. Then $(p+q)R \cap Q\neq 0$, so $(p+q)R\cap N\neq 0$ where $(p+q) ann_R(p)\neq 0$. Now assume that $(p+q)ann_R(p)=0$. Then $ann_R(p)\subseteq ann_R(p+q)$, i.e., $ann_R(p+q)= \{r\in ann_R(p) \backslash (p+q)r=0\}$ where $pr=0$. Therefore by theorem (1.1.9(3)) $S(p+q)\subseteq Sp$. But $p+q\in Sp\subseteq p$, since p is fully invariant, then $p+q\in P$, but $N\subseteq P$, hence $p+q\in N$ implies that $(p+q)R\cap N\neq 0$.

We also need the following definition.

Definition 2.2.2 [15] : A submodule A of an R-module M is said to be closed submodule of M if A has no proper essential extension in M, i.e., if $A\underset{e}{\leq}B \leq M$, then B=A .

For example $\{ \bar{0}, \bar{3}\}$closed in Z_6 , $\{ \bar{0}, \bar{2}, \bar{4}\}$ closed in Z_6 , but $\{ \bar{0}, \bar{2}\}$ is not closed in Z_4.

Recall that every non-zero submodule N of M_R has (by Zorn's lemma **[10,p.25]**) a maximal essential extension P in M called closure of N in M.

Theorem 2.2.3 [15] : Let M_R be a P.Q.-injective module and suppose a non-zero submodule N of M has a fully invariant closure P in M. Then P contains every essential extension of N, so P is the unique closure of N in M.

Proof : Suppose $N\subseteq^{ess}Q\subseteq M$. then by lemma (2.2.1) $N\subseteq^{ess}P+Q$. Since $N\subseteq P$, then $P\subseteq^{ess}P+Q$, but P is closed, this means that P=P+Q, so $Q\subseteq P$. The result follows.

Our main concern here is with uniform submodules U of a module M_R. By Zorn's lemma **[10, p.25]** . U has maximal uniform extensions in M. These are all the closure of U, in fact, they are precisely the closed uniform submodules of M. So every uniform closed submodule is a maximal uniform submodule and is a maximal uniform extension of each of its non-zero submodules.

Remark 2.2.4 [15] : If U is a uniform submodule of M with $S=end(M_R)$, define $AU=\{\alpha\in S \backslash \ker(\alpha)\cap U\neq 0\}$. If $uR\neq 0$ is cyclic uniform, we call u a uniform element of M and write AuR=AU. It can be easily checked that AU is a left ideal in S .

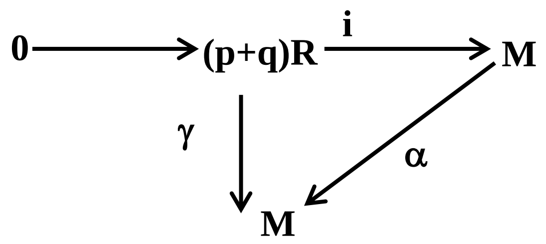
Proposition 2.2.5 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$. If u is a uniform element of M, then Au is the unique maximal left ideal of S containing $ann_S(u)$.

Proof: Suppose that $ann_S(u)\subset X$ where X is a left ideal of S, $X\neq S$. Now if $\alpha\in X-Au$, then $\ker(\alpha)\cap uR=0$, hence by proposition (1.1.10) $S=ann_S(o)=ann_S[\ker(\alpha)\cap uR]=S\alpha+ann_S(u)\subseteq X$. Then $S\subseteq X$ a contradiction. Thus $X \subseteq Au$, also Au is unique because it is maximal and $X\subseteq Au$. Thus the proof is complete.

Corollary 2.2.6 [14] : Let R be a P-injective ring. If $u\in R$ is a uniform element, define $M_u=\{x\in R\backslash ann(x)\cap uR\neq 0\}$. Then Mu is the unique maximal ideal which contains ann(u) .

Proposition 2.2.7 [15] : Let M_R be a P.Q.-injective module and let P and Q be fully invariant maximal uniform submodules of M. If Ap=AQ then P=Q.

Proof : It suffices (by theorem 2.2.3) to show that $P\cap Q\neq 0$,since then both P and Q are fully invariant closures of $P\cap Q$. Assume on the contrary that $P\cap Q=0$. Choose $0\neq p\in P$, $0\neq q\in Q$ and consider $\gamma : pR + qR \rightarrow M$ given by γ (pr+qs)=pr where r,s $\in R$. It is easily seen that γ is well-defined. Since M is P.Q.-injective , then there exists $\alpha\in S$ which extends γ.



We have $\alpha(p)=\alpha[i(p+o)]= \gamma(p+o)=p$, hence $\alpha(p)=p$, so $p-\alpha(p)= (1-\alpha)(p)=0$, then $p\in\ker(1-\alpha)$. Also $\alpha(q)=\alpha[i(o+q)]=\gamma(o+q)=0$, hence $\alpha(q)=0$, then $q\in\ker(\alpha)$. Thus $Ap=\{1-\alpha\in S\backslash\ker(1-\alpha)\cap P\neq 0\}$, $AQ=\{\alpha\in S\backslash\ker(\alpha)\cap Q\neq 0\}$. So $1-\alpha\in Ap$ and $\alpha\in AQ=Ap$. It follows that $1\in Ap$ a contradiction. Hence $P\cap Q\neq 0$ and P=Q .

Proposition 2.2.8 [15] : Let M_R be a P.Q.-injective module with $S=end(M_R)$ and let $N=u_1R\oplus\ldots\oplus u_nR$ where each $u_i\in M$ is a uniform element. If $A\subseteq S$ is a maximal left ideal not of the form AU for any uniform submodule $U\subseteq M$, then there exists $\beta\in A$ such that $\ker(1-\beta)\cap N \subseteq^{ess} N$.

<u>Proof</u> : Since $A\neq Au_1$, let $\ker(\alpha)\cap u_1R=0$ where $\alpha\in A$. Then $ann_R(\alpha u_1)\subseteq ann_R(u_1)$, in fact, let $r\in ann_R(\alpha u_1)$, then $\alpha(u_1)r=0=\alpha(u_1r)$, implies that $u_1r\in\ker(\alpha)\cap u_1R$, hence $u_1r=0$, thus $r\in ann_R(u_1)$ and so $u_1\in S\alpha u_1$ by theorem (1.1.9(3)), say $u_1=\beta\alpha u_1$ where $\beta\in S$, so $u_1-\beta\alpha u_1=(1-\beta_1)u_1=0$ where $\beta_1=\beta\alpha\in A$. If $\ker(1-\beta_1)\cap u_iR\neq 0$ for each $i>1$, we are done. Since $(1-\beta_1)u_2R\cong u_2R$, if $\ker(1-\beta_1)\cap u_2R=0$, then $(1-\beta_1)u_2$ is a uniform element and so, as before, there exists $\gamma\in A$ such that $(1-\gamma)(1-\beta_1)=0$, i.e., $\ker(1-\gamma)\cap u_2R=0$ where $\gamma\in A$. Then $ann_R(1-\gamma)u_2\subseteq ann_R(u_2)$, implies that $u_2\in S(1-\gamma)u_2$, so $u_2=\beta_1(1-\gamma)u_2$ where $\beta_1\in S$, hence $u_2-\beta_1(1-\gamma)u_2=0=(1-\beta_1)(1-\gamma)u_2$. If we take $\beta_2=\gamma+\beta_1-\gamma\beta_1$, then $\beta_2\in A$ and have $(1-\beta_2)u_2=0$ and $(1-\beta_2)u_1=0$. This means that $\ker(1-\beta_2)\cap u_iR\neq 0$ for i, 1, 2. This process continues to give $\beta\in A$ such that $\ker(1-\beta)\cap u_iR\neq 0$ for each i, this complete the proof.

Section 2.3 Quasi-Principally - Injective Modules and Continuous Modules :

An R-module M with C_1-condition and C_2-condition is called continuous module where M is said to have the C_1-condition, if every submodule of M is essential in a direct summand of M **[8]** and it has C_2-condition if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. In this section we study the relation between Q.P.-injective modules and continuous modules. For a reference on continuous module see **[8]** .

Proposition 2.3.1 [8] : If M is uniform and P.Q.-injective module, then M is continuous module.

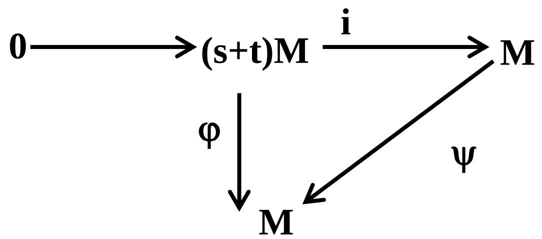
Proof : Since M is uniform, then every submodule of M is essential in M, hence M has C_1-condition. Also M is P.Q.-injective module, then by proposition (1.2.13) M has C_2-condition. Hence M is continuous module.

Definition 2.3.2 [20] : A submodule K of an R-module M is called M-cyclic submodule of M if it is isomorphic to M/X for some submodule X of M. Equivalently, K is M-cyclic if there exists $\alpha \in end(M)$ such that $K=\alpha(M)$.

Proposition 2.3.3 [8] : Let M be a Q.P.-injective module. If $S=end(M_R)$ is local, then for any non-zero fully invariant M-cyclic submodules A and B of M, $A \cap B \neq 0$.

Proof : Let $0\neq s(M)=A$, $0\neq t(M)=B$ where s, $t\in S$ and $A\cap B=0$. Define the map $\varphi : (s+t)(M) \to M$ by $(s+t)(m) \mapsto s(m)$ for every $m\in M$. This map is well-defined , in fact , $(s+t)(m) = (s+t)(m')$ implies $s(m-m') = t(m'-m) \in A \cap B=0$, so $s(m)=s(m')$ where m, $m'\in M$. Since M is P.Q.-injective, then there exists $\psi \in S$ such that

$(\Psi oi)|_{(s+t)(M)} = \Psi|_{(s+t)(M)} = \varphi|_{(s+t)(M)}$, i.e., for any m∈M , φ(s+t) (m)=ψ(s+t) (m). Since φ (s+t) (m) = s(m), then s=ψ(s+t). This implies that s-ψ(s+t)= s-ψ_s=$(1-\psi)_s$= ψ_t. . Since A and B are fully invariant submodules, then $(1-\psi)_{s(M)}$ ⊆A and $\psi_{t(M)}$⊆B. Since $(1-\psi)_{s(M)}$= $\psi_{t(M)}$∈A∩B=0 , then $(1-\psi)_s$=0 and ψ_t=0 . But S is local, then either ψ or 1-ψ is invertible, if ψ is invertible, then t=0 a contradiction or 1-ψ is invertible, then s=0 a contradiction. Hence A∩B≠0 .



Corollary 2.3.4 [8] : If M is a Q.P.-injective duo module which is a self-generator with local endomorphism ring, then M is uniform, hence it is continuous .

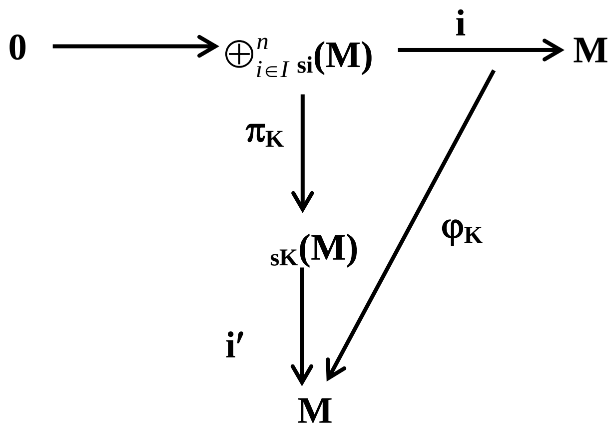
Proof : Since M is self-generator, then for any m∈M, mR contains a non-zero M-cyclic submodule. Hence by proposition (2.3.3) M is uniform. Since M is P.Q.-injective, then by proposition (2.3.1) M is continuous.

It is known that every multiplication module is duo and self-generator **[17]**, thus

Corollary 2.3.5 : If M is Q.P.-injective multiplication module with S=end(M_R) is local, then M is uniform, hence M is continuous.

Proposition 2.3.6 [8] : Let M be a Q.P.-injective and $\oplus_{i\in I}B_i$ a direct sum of fully invariant M-cyclic submodules of M. Then for any fully invariant submodule A of M A∩($\oplus_{i\in I}B_i$) =$\oplus_{i\in I}$(A∩B_i) .

Proof : It is known that $\oplus_i(A\cap B_i) \subseteq A\cap (\oplus_i B_i)$. Let $a \in A\cap(\oplus_i B_i)$ and $B_i=s_i(M)$ such that $s_i\in S=end(M_R)$ .Then $a=b_1+b_2+\ldots+b_n$ where $b_i=s_i(m_i)\in B_i$ for some $m_i\in M$. Notice that the $\sum_{i\in I}S_{si}$ is direct. Since B is fully invariant, then $S_{si}(M)=S_{Bi}\subset B_i$ $(i\in I)$. So let π_K: $\oplus_{i=1}^{n} S_i$ (M) $\rightarrow$ S_K (M),$1\leq k\leq n$ be the projection. Since M is Q.P.-injective , then by **[18],** we can find an endomorphism φ_K: M $\rightarrow$ M which extends π_K . Since π_K is onto, then there exists $b_K\in S_K$ (M) such that $b_K= \pi_K(a) \subseteq A\cap B_K$ for any $1\leq k\leq n$ because A is fully invariant. Hence $A\cap(\oplus_{i\in I} B_i) \subset \oplus_{i\in I}(A\cap B_i)$.



Proposition 2.3.7 [8] : Let M be a Q.P.-injective and duo module. If A and B are direct summands of M, then so are $A\cap B$ and A+B .

Proof : Let $M=A \oplus A_1=B \oplus B_1$. Then by proposition (2.3.6) we have $B=B\cap M=B\cap(A\oplus A_1)=(B\cap A)\oplus(B\cap A_1)$. Hence $M=(B\cap A) \oplus (B\cap A_1)\oplus B_1$. Thus $A\cap B$ is a direct summand of M. More over $A+B=A+(B\cap A) \oplus (B\cap A_1)= [A+(B\cap A)] \oplus (B\cap A_1)=A+(B\cap A_1)$. Since M is Q.P.-injective, A and B are direct summands of M, then by **[18]** M has C_3-condition, hence A+B is a direct summand of M and the proof is now complete.

Corollary 2.3.8 : Let M be a Q.P.-injective multiplication module. If A and B are direct summands of M, then so are $A\cap B$ and A+B .

We observed that every uniform, Q.P.-injective module is continuous , we now consider the case when M is Q.P.-injective module which is a direct sum $M=\oplus_{i\in I}M_i$ of uniform submodules. In this case, if M is duo, then by proposition (2.3.6) every submodule A of M can be written in the form $A=\oplus_{j\in J}(A\cap M_j)$ where $J\subset I$ and $A\cap M_j \neq 0$, $j\in J$. Since each $A\cap M_j \leq_e M_j$,we see that $A\leq_e \oplus_{j\in J} M_j$. Thus we have proved.

Theorem 2.3.9 [8] : Let $M=\oplus_{i\in I} M_i$ be a Q.P.-injective module where each M_i is uniform. If M is duo module, then M is continuous module .

Before the next result we need this defintion.

Definition 2.3.10 [10, p.275] : An R-module M is called semiperfect if every epimorphic image of M has a projective cover where an epimorphism $\sigma:P \rightarrow M$ is called projective cover of M if P is projective and σ is small epimorphism.

Theorem 2.3.11 [8] : Suppose that M is semiperfect, duo, Q.P.-injective module. If M is a self-generator, then M is continuous module.

Proof : Since M is Q.P.-injective module, then by proposition (1.2.13) M has C_2-condition. Hence it is enough to prove that M has C_1-condition. Since M is self-generator and semiperfect, then by **[19, 42.5]** we can write $M=\oplus_{i\in I} M_i$ where M_i/rad (M_i) is simple for each $i\in I$. then rad(M_i) is maximal in M_i. Since each M_i is self-generator and semiperfect, rad (M_i) is small in M_i and hence M_i is indecomposable. By **[10, p.285]**, end(M_i) is local for each $i\in I$. By proposition (2.1.5) each M_i is duo and a self-generator. Since any direct summand of P.Q.-injective is again P.Q.-injective, then by corollary (2.3.4) each M_i is uniform, then M has C_1-condition. Therefore M is continuous.

<u>Corollary 2.3.12 :</u> Suppose that M is semiperfect. If M is Q.P.-injective multiplication module, then M is continuous module.

CHAPTER THREE

SEMI – INJECTIVE MODULES AND FULLY STABLE MODULES

Introduction:

Let M be an R-module with S=end(M_R). In this chapter we study briefly notions of injectivity, like M-principally injective, semi-injectivity, π- injectivity and direct- injectivity. More over, we study the notions of fully stability. This chapter consists of two sections.

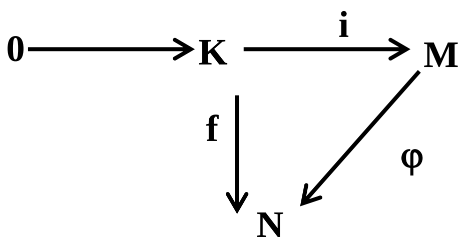
In section 1, we study the above mentioned types of injectivity and we study the definition of M-cyclic submodule instate of cyclic submodule, this concept is studied in **[20]** .

In section 2, we study fully stable and fully invariant modules in principally quasi-injective modules and rings.

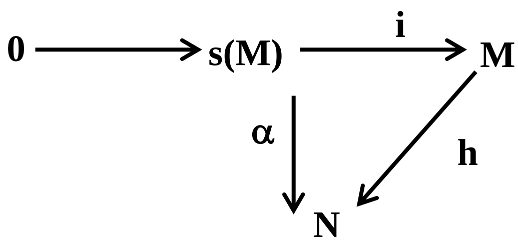
Section 3.1 On the Endomorphism Ring of a Semi - Injective Module

In this section we study the endomorphism rings of semi-injective modules, in particular we study the Jacobson radical of S with its relation to the sets W(S) = $\{s\in S \backslash \ker(s) \leq_e M\}$ and $\Delta=\{s\in S \backslash \ker(1+ts)=0$ for all $t\in S\}$. Most of the results of this section appeared in **[20].**

Definition 3.1.1 [20] : An R-module N is called M-principally injective if every R-homomorphism from M-cyclic submodule K of M to N can be extended to M, in general, the following diagram is commutative, $\varphi_o i=f$ where $K\cong M/L$ and L is a submodule of M.



Equivalently, for any endomorphism s of M, every homomorphism from s(M) to N can be extended to a homomorphism from M to N, $h_o i=\alpha$



Remark 3.1.2 : An M-cyclic submodules and cyclic submodules are completely different concepts.

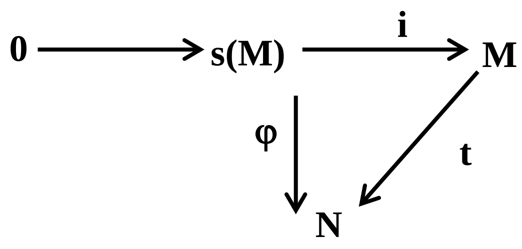
For example, Z as a submodule of the Z-module Q is cyclic but not Q-cyclic because every non-zero homomorphism $f:Q \rightarrow Q$ is an epimorphism. On the other hand, let $M=Z_2\oplus Z_2\oplus Z_3$ considered as a

Z-module. Since $M/Z_3 \cong Z_2 \oplus Z_2$, then $Z_2 \oplus Z_2$ is M-cyclic. But $Z_2 \oplus Z_2$ is not cyclic .

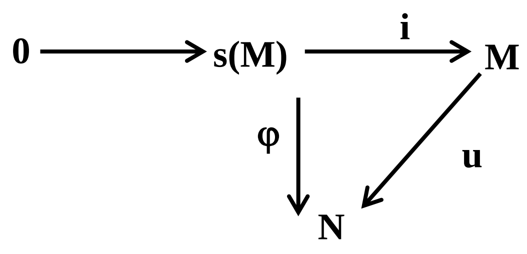
The following proposition gives a characterization of M-principally injective modules.

Proposition 3.1.3 [20] : Let M and N be R-modules. Then N is M-principally injective if and only if for each $s \in S = end\ (M)$. $Hom_R\ (M,N)_S = \{f \in Hom_R\ (M,N) \setminus f(ker\ (s)) = 0\}$ where $Hom\ (M,N)_S = \{fs \setminus f \in Hom\ (M,N)\}$.

Proof ⇒) Assume N is M-principally injective, we want to show that $Hom_R\ (M,N)_S = \{f \in Hom_R\ (M,N) \setminus f(ker\ (s)) = 0\}$, it is clear that $Hom_R\ (M,N)_S \subseteq \{f \in Hom_R\ (M,N) \setminus f(ker\ (s)) = 0\}$. Conversely, let $f \in Hom_R\ (M,N) \setminus f(ker\ (s)) = 0$, hence $ker\ (s) \subseteq ker\ (f)$. Define a homomorphism $\varphi: s(M) \to N$ by $\varphi[s(M)] = f(m)\ \forall\ m \in M$, φ is well-defined, in fact, let $s(m_1) = s(m_2)$ where $m_1, m_2 \in M$, hence $s(m_1) - s(m_2) = s(m_1 - m_2) = 0$, then $m_1 - m_2 \in ker\ (s) \subseteq ker\ (f)$, so $m_1 - m_2 \in ker\ (f)$. Thus $f(m_1 - m_2) = f(m_1) - f(m_2) = 0$, which implies $f(m_1) = f(m_2)$, so $\varphi\ [s(m_1)] = \varphi\ [s(m_2]$. Since N is M-principally injective, then there exists an R-homomorphism $t: M \to N$ such that $t_o i = \varphi$ where $i: s(M) \to M$ is the inclusion map. Now $f(M) = \varphi[i(s(M))] = \varphi[s(M)] = t[i(s(M))] = t[s(M)]$. Hence $f = ts$ and therefore $f \in Hom_R(M,N)_S$



⇐) Suppose that $\varphi: s(M) \to N$ is an R-homomorphism. Then $\varphi_s \in Hom_R(M,N)$ and $\varphi_s\ [ker\ (s)] = 0$. By assumption, we have φ[s(M)]= u[s(M)] = u[i(s(M))] for some u ∈ $Hom_R(M,N)$. This shows that N is M-principally injective.



<u>Definition 3.1.4 [20] :</u> An R-module M is called semi-injective if it is M- principally injective module.

It was shown lemma (1.2.14) that W(S) ⊆ J(S), the following gives a condition that implies equality.

<u>Proposition 3.1.5 [20]</u> : Let M be semi-injective , then W(S) ⊆ J(S) and equality holds if S/W(S) is regular.

<u>Proof :</u> If s∈J(S), then 1-sα has a left inverse. Since S/W(S) is regular, then s+W(S) = sαs + W(S) for some α∈S. This implies that s-sαs=(1-sα) s∈W(S), so there exists g∈S such that g(1-sα)s=1.s = s ∈W(S). This shows that W(S)=J(S) .

<u>Corollary 3.1.6 [20]</u> : : Let M be semi-injective . If S/J(S) is regular, then S/W(S) is regular if and only if J(S)=W(S).

<u>Proof ⇒)</u> Since S/W(S) is regular, then by proposition (3.1.5) J(S)=W(S).

<u>⇐)</u> Since S/J(S) is regular and J(S)=W(S), then S/J(S)=S/W(S) is regular.

Remark 3.1.7 [20]: Let M be semi-injective ,then J(S)=Δ

Proof : Let s ∈J(S), then for each t∈S, 1+ts has a left inverse in S, there exists g∈S such that g(1+ts)=1_M, hence 1+ts is monomorphism, thus ker (1+ts)=0. Therefore s ∈Δ. On the other hand, if s∈Δ, then ker (1+ts)=0 for all t∈S, f[ker(1+ts)]=f(0)=0, which implies that s∈ann_S[ker (1+ts)]=S, thus by proposition (3.1.3) s=s(1+ts). In particular g(1+ts)=1_M for some g∈S, then by **[10, p.220]** s∈J(S) .

Remark 3.1.8 [20] : Let M be semi-injective . If M is uniform, then Z(Ss) ⊆ J(S).

Proof : Let s∈Z(S), then ker (s) ≠ 0. For any t∈S, we have ker (s) ∩ ker (1+ts)=0, then ker (1+ts)=0 . Hence by (3.1.7) s ∈J(S).

Before the next result we need some definitions.

Definition 3.1.9 [20] : An R-module M is called π-injective if for all submodules U and V of M with U∩V=0, there exists f∈S with U⊆ker f and V⊆ker (1-f) .

Definition 3.1.10 [20] : An R-module M is said to be direct-injective if for any direct summand D of M, every monomorphism f: D → M splits.

Theorem 3.1.11 [20] : Let M be a semi-injective R-module. Then

(1) If S is local then J(S)= {s∈S\ ker (s) ≠ 0}.

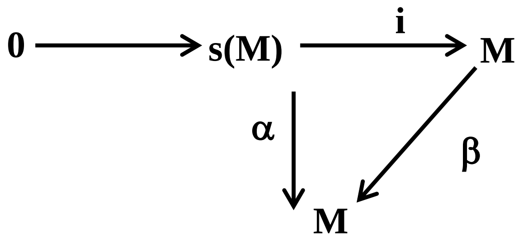
(2) If Im(s)$\subseteq^{ess}$M where s∈S , then any monomorphism t: s(M) → M can be extended to a monomorphism in S .

(3) If M is uniform, then S is local ring and J(S)=W(S) .

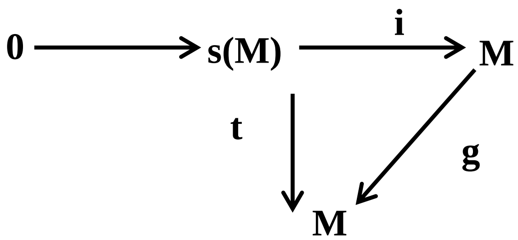
(4) For s∈S, if M is uniform and s is left invertible, then s is invertible.

(5) M is uniform if and only if S is local and M is π-injective .

Proof : (1) Since S is local, $Ss \neq S$ for any $s \in J(S)$. If ker (s)=0, then α : $s(M) \rightarrow M$ given by $\alpha[s(m)]=m$ for any $m \in M$ is well-defined and R-homomorphism. Since M is semi-injective, there exists $\beta \in S$, extension of α. $\beta[s(m)[=\beta[i(s(m))]=\alpha[s(m)]=m$, hence $\beta_s=1_M$ such that $\beta \in S$, so Ss=S, which is a contradiction. This shows that $J(S)=\{s \in S \backslash \ker(s) \neq 0\}$. The other inclusion $\{s \in S \backslash \ker(s) \neq 0\} \subseteq J(S)$ always holds .



(2) Since M is semi-injective, then there exists $g \in S$ such that $g[s(m)]=g[i(s(m))]=t[s(m)]$where $m \in M$. Thus $Im(s) \cap \ker(g)=0$, in fact, if $x \in \ker(g) \cap Im(s)$, then $x \in \ker(g)$ and $x \in Im(s)$. This implies that g(x)=0 and there exists $y \in M$ such that x=s(y). Thus $0=g(x)=g[s(y)] =t[s(y)]=t(x)=0$, hence $x \in \ker(t)$. Since t is monomorphism, then x=0. Thus by definition (1.1.15) ker (g)=0, which implies that g is monomorphism.



(3) Since M is direct-injective, S is local provided that M is uniform **[19, 41.22]**. It follows that J(S)=W(S) by (1) .

(4) Since s has a left inverse, then there exists $f \in S$ such that $fs=1_M$, note that f is onto and s is 1-1, hence ker (s)=0, but M is uniform, then by (3) S

is local and by(1) ker (s) $\neq$ 0, then we have s$\in$J(S), this implies that 1-s $\in$ J(S), so s is invertible .

(5) Since M is uniform, then by (3) S is local, so M is π-injective. Conversely, let U and V be submodules of M such that U$\cap$V=0. Since M is π-injective, then there exists f$\in$S such that U$\subset$ ker (f) and V$\subseteq$ker (1-f). But S is local, then either f or 1-f belongs to J(S). If f$\in$J(S), then g(1-f)=1 for some g$\in$S. Thus ker (1-f)=0, implies that V=0. Other wise U=0. Hence M is uniform .

Proposition 3.1.12 [20]: Suppose M is a semi-injective and π-injective module. If S is semiperfect, then M=$\oplus_{i=1}^{n}$ U_i where U_i is uniform and semi-injective for each i .

Proof : Since S is semiperfect and M is semi-injective, then M= $U_1\oplus\ldots\oplus U_n$ where each end (U_i) is local. Note that U_i is semi-injective. So by **[19, 41.20]** each U_i is π-injective. Thus by proposition (3.1.11(5)) we see that U_i is uniform.

Proposition 3.1.13 [20] : If Soc(M)$\subseteq^{ess}$M, then

(1) W(S)=ann_S(Soc(M)) .

(2) S/W(S) is embeded in end_R(Soc(M)) as a subring.

Proof : (1) Let s$\in$W(S), then ker (s) $\subseteq^{ess}$M, so by definition of (1.1.17) Soc(M) $\subseteq$ ker (s), hence s(Soc(M))=0. Which implies that s$\in ann_S$ (Soc (M)). On the other hand, let s $\in ann_S$ (Soc(M)) where s$\in$S, then s(Soc(M))=0, hence Soc(M) $\subseteq$ ker (s), so Soc(M) $\subseteq^{ess}$ M. thus ker (s) $\subseteq^{ess}$ M and s$\in$W(S).

(2) For each s$\in$S, let φ(s): Soc(M) $\rightarrow$ Soc(M) be defined by $(\varphi(s)_{(x)}$ =s(x). Since Soc(M) is fully invariant in M, then φ(s) $\in end_R$ (Soc(M)) and φ:S $\rightarrow$ end_R(Soc(M)) is a ring homomorphism. Note that ker (φ)= W(S), in

fact, $s \in W(S) \Leftrightarrow \ker(s) \subseteq^{ess} M$, i.e., $\forall\ 0 \neq m \in M$, there exists $\alpha \in S$ such that $\alpha(m) \neq 0$ and $\alpha(m) \in \ker(s)$, hence $s(\alpha(m))=0$, implies that $\varphi[s(\alpha(m))]=0$. Thus $s \in \ker(\varphi)$, therefore by first isomorphism theorem **[10, p.56]** $S / W(S) \cong end_R(Soc(M))$.

Proposition 3.1.14 [20] : If M is semi-injective and a self-generator and if $Soc(M) \subseteq^{ess} M$, then

(1) $J(S)=ann_S(Sco(M))$.

(2) $S/J(S) \cong end_R\ (Soc(M))$.

Proof : (1) Since M is semi-injective and a self-generator, then by **[18]** $J(S)=W(S)$. Thus by proposition (3.1.13) $J(S)=ann_S(Soc(M))$.

(2) Since M is semi-injective, every R-homomorphism in $end_R\ (Soc(M))$ can be extended to an R-homomorphism in S. Then by (1) and proposition (3.1.13(2)) $S/J(S)$ is isomorphic to $end_R\ (Soc(M))$ as a ring.

Since every projective module is self-generator, then we have

Corollary 3.1.15 : If M is semi-injective and projective and if $Soc(M) \subseteq^{ess} M$, then

(1) $J(S)=ann_S(Soc(M))$.

(2) $S/J(S) \cong end_R\ (Soc(M))$.

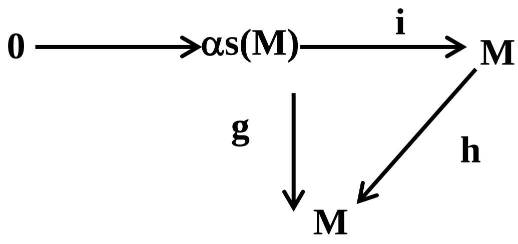
Proposition 3.1.16 [20] : Let M be a semi-injective R-module.

(1) If Im(s) is a simple right R-module where $s \in S$, then Ss is a simple left S-module.

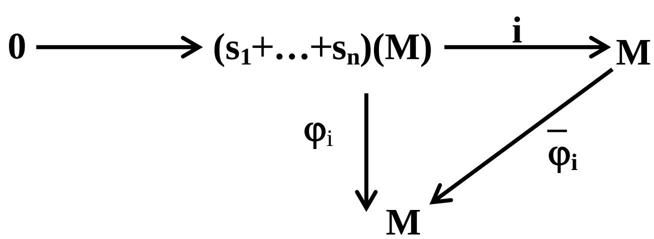
(2) If $s_1(M) \oplus \ldots\ldots \oplus s_n(M)$ is direct where $s_1, s_2, \ldots, s_n \in S$ then $S(s_1+\ldots+s_n)=Ss_1+\ldots+Ss_n$.

Proof : (1) Let A be a non-zero submodule of Ss and $0 \neq \alpha s \in A$. then $S\alpha s \subset A$. Since Im(s) is simple, $\ker(g) \cap Im(s)=0$. Define $g: \alpha s(M) \to M$

by $g[\alpha s(m)]=s(m)$ for every $m\in M$. It is obvious that g is an R-homomorphism. Since M is semi-injective, then there exists a homomorphism $h \in S$ such that $h(\alpha s)=g(\alpha s)$. Therefore $h(\alpha s)=s$, $s\in S\alpha s$. It follows that $S\alpha s = Ss$ and $A=Ss$.



(2) Let $\alpha_1 s_1+\ldots+\alpha_n s_n \in Ss_1+\ldots+Ss_n$. For each i, defined $\varphi_i:(s_1+\ldots+s_n)(M) \to M$ by $\varphi_i\ [s_1+\ldots+s_n(m)]=s_i(m)$ for every $m\in M$. Since $s_1(M)\oplus\ldots\ldots\oplus s_n(M)$ is direct, φ_i is well-defined, so it is clear that φ_i is an R-homomorphism. Then there exists an R-homomorphism $\bar{\varphi}_i \in S$ which is an extension of φ_i. Then $s_i= \varphi_i\ (s_1+\ldots+s_n) = \bar{\varphi}_i\ (s_1+\ldots+s_n) \in S(s_1+\ldots+s_n)$ for every $i=1,2,\ldots,n$. Consequently, $\alpha_1 s_1+\alpha_2 s_2+\ldots+\alpha_n s_n \in S(s_1+\ldots+s_n)$. Hence $Ss_1+\ldots+Ss_n\subset S(s_1+\ldots+s_n)$. The other inclusion always holds.



<u>Proposition 3.1.17 [20]</u> : Every duo and semi-injective module has the (SIP) and (SSP) .

Proof : Write $M=s(M)\oplus K$ and $M=t(M)\oplus L$ where K, L are submodules of M. Since M is duo, $s(M)=s(t(M)\oplus L)=st(M)+s(L)\subset(s(M)\cap t(M))+(s(M)\cap L)=(s(M)\cap t(M))\oplus(s(M)\cap L)\subset s(M)$. Then $s(M)\cap t(M)$ is a direct summand of M. Now we write $M=s(M)\cap t(M)\oplus N$. Then $t(M)=t(M)\cap(s(M)\cap t(M)\oplus N) = s(M)\cap t(M)\oplus t(M)\cap N$, so $s(M)+t(M) =s(M)+(s(M)\cap t(M)\oplus t(M)\cap N)=s(M)+t(M)\cap N=s(M)\oplus t(M)\cap N$. Since $s(M)$ and $t(M)\cap N$ are direct summands, $s(M)+t(M)$ is a direct summand of M by C_3-condition.

Definition 3.1.18 [20] : A ring R is called semi-regular if R/J(R) is regular and idempotents can be lifted module J(R) .

Remark 3.1.19 [20] : R is semi-regular if and only if for each element $a\in R$, there exists $e^2=e\in Ra$ such that $a(1-e)\in J(R)$.

Remark 3.1.20 [20] : For every $s\in S/J(S)$, there exists a non-zero idempotent $\alpha\in Ss$ such that $\ker(s)\subset\ker(\alpha)$ and $\ker[s(1-\alpha)]\neq 0$.

Theorem 3.1.21 [20] : For a semi-injective module M, if S is semi-regular, then (3.1.20) holds.

Proof : Let $s\in S/J(S)$. Then there exists $\alpha^2=\alpha\in Ss$ such that $s(1-\alpha)\in J(S)$. Then $\alpha\neq 0$ and $\ker(s)\subset\ker(\alpha)$. If $\ker[s(1-\alpha)]=0$, then $gs(1-\alpha)=1_M$ for some $g\in S$ by the semi-injectivity of M . It follows that $\alpha=0$, a contradiction. Hence $\ker[s(1-\alpha)]\neq 0$.

Section 3.2 Fully Stable Modules

Recall that a submodule N of an R-module M is said to be fully invariant if $f(N)\subseteq N$ for each endomorphism f of M **[21]**, we call M invariant if each of its submodules is fully invariant. Recall that a submodule N of an R-module M is said to be stable if $f(N)\subseteq N$ for each $f\in Hom(N,M)$, and the module M is said to be fully stable if each submodule is stable **[1]**. In this section we study these notions in P.Q.-injective rings and fully stable

Remark 3.2.1 : It is clear that each fully stable module is fully invariant, but the converse is not true .

For example, Z as a Z-module is fully invariant, but it is not fully stable.

Note 3.2.2 : R is fully stable if and only if R is fully stabel as R-module .

Remark 3.2.3[1] :

(1) An R-module M is fully stable if and only if every cyclic submodule is stable.

(2)It is known that an R-module M is fully stable if and only if $ann_M(ann_R(x)) = xR\ \forall\ x\in M$.

A statement similar to the following statement is known for pointwise injective modules **[3]**, we prove it for P-injective rings .

Proposition 3.2.4 : Let R be a ring, then R is P-injective if and only if R is fully stabel.

Proof : By corollary (1.1.12(2)) $ann_R ann_R(x)=Rx\ \forall\ x\in R$, then by the last remark. R is fully stable.

Recall that a submodule N of an R-module M is said to satisfy Baer criterion if for every R-homomorphism $\varphi:N \to M$, there exists an element r in R such that $\varphi(n)=rn$ for each n in N [1] .

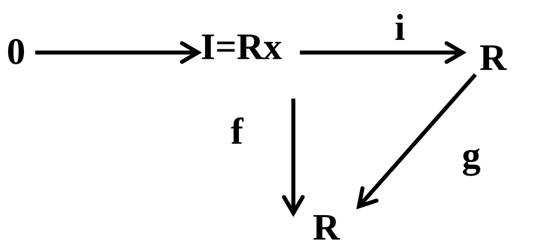
Notice that the concepts of Baer condition and Baer criterion coincide for rings.

Clearly, every module which satisfies Baer criterion is fully stable.

The following result shows the relation between P-injective rings and Baer condition.

Proposition 3.2.5 : Let R be a P-injective ring, then R satisfies Baer's condition for every principal ideal I of R.

Proof : Let $I=Rx$, $x\in R$ and let $f:Rx \to R$ be any R-homomorphism. Consider the following diagram



Since R is P-injective, then there exists $g: R \to R$ such that $g_o i = f$. Now $\forall$ $t\in I$, $t=rx$ where $r\in R$, $f(t)=f(rx)=(g_o i)_{(rx)}=g[i(rx)]=g(rx) =rg(x) =rx\, g(1)=t\, g(1)$, take $g(1)=y$, hence $f(t)=ty$ where $y \in R$.

Proposition 3.2.6 : If R is P-injective ring, then for each ideals I and J in R with I+J is principal, $ann_R(I\cap J)=ann_R(I)+ann_R(J)$.

Proof : Let $x\in ann_R(I\cap J)$. Define $f : I+J \to R$ by $f(a+b)=bx$ where $a\in I$, $b\in J$, f is well-defined. Since R is P-injective, then there exists $y\in R$ such that $f(a+b)=(a+b)\, y=bx$. In particular $0=f(a)=ay$ holds $\forall$ $a\in I$, this implies that $y\in ann_R(I)$. $\forall$ $b\in J$, $f(b)=by=bx$, so $bx-by=b(x-y)=0$, hence $x-y\in ann_R(J)$. Now $x=y+x-y \in ann_R(I)+ann_R(J)$.

The converse of proposition (3.2.6) is not true.

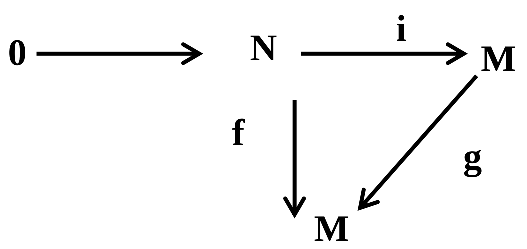
For example, let nz, mz be ideals in Z, $ann_Z(nZ \cap mZ)=ann_Z(nmZ)=0$, also ann(nZ)+ann(mZ)={0}, but Z is not P-injective.
Now we raise the following question: Is every P.Q.-injective module fully stable module?

The answer is No.
For example, Q as a Z-module is injective, hence P.Q.-injective, but Q as a Z-module is not fully stable.

However, we have the following

Proposition 3.2.7 : Let M be a multiplication R-module. If M is P.Q.-injective module , then M is fully stable.

Proof : It is enough to show that every cyclic submodule is stable. Let N be a cyclic submodule of M, let f: N $\rightarrow$ M be any R-homomorphism. Sinec M is multiplication, then N=IM for some idael I of R. Thus for each $n \in N$, $n=\sum_i r_i m_i$, $r_i \in I$, $m_i \in M$. Consider the following diagram, $g_o i=f$.



$\forall\ n \in N$, $f(n)=(g \circ i)(n)=g(n)=g(\sum_{i=1}^{n} r_i m_i) = \sum_{i=1}^{n} r_i g(m_i) \in IM=N$ where $r_i \in I$, $m_i \in M$. Hence N is stable,then by remark (3.2.3) M is fully stable

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On Principally Quasi-Injective Modules and Semi-Injective Modules

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By

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ABSTRACT

Let M be an R-module with endomorphism ring S. The module M_R is called principally quasi-injective, if every R-homomorphism from any cyclic submodule of M to M can be extended to an endomorphism of M. An R-module N is called M-principally injective, if every R-homomorphism from M-cyclic submodule K of M to N can be extended to M. An R-module M is called semi-injective if it is M-principally injective.

These concepts were studied by Nicholson, Yousif and wangwal. The main purpose of this thesis is to study principally quasi-injective modules and semi-injective modules. We give the details of proofs of known results, supply some example, and add few new results.

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