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Construction of Sylvester-Hadamard Matrices by Using Binary Code

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Abstract: A simple method is presented which defines Sylvester-Hadamard matrices in terms of products of binary code. This method is based representation of natural number as binary code which take only two value 0 or 1. Such a Hadamard matrices generator can be used to find the spectral coefficients of Boolean functions.

Keywords: Hadamard matrix, binary code, Kroncker product, Boolean function

Introduction

Hadamard matrices were defined by the French mathematician M.j. Hadamard in 1893, [1] called now Hadamard matrices. These matrices contain only the entries+1 and -1. They are used in many applications like, signal processing, optical multiplexing, error correction coding and design and analysis of statistics, [2]. Also, Oliver Hunt, [3] used them in Image coding. In communication system, digital image processing and orthogonal spreading sequences, Hadamard matrices are used for direct sequences spread spectrum code division multiple access, [4].

Hadamard matrix of order N=2p, (p is positive integer) used to encode the mask: m=Hx, where H is the Hadamard matrix or Sylvester matrix, x is the single wevelength intensity matrix, [5].

Sylvester-Hadamard Matrix

A square matrix with elements ±1, whose distinct row vectors are orthogonal is an Hadamard matrix of order N=2p, p is positive integer:

$$H_1 = [1]$$ ... (2.1)

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ ... (2.2)

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$ … (2.3)

The Sylvester-Hadamard matrix of order N=2p, is generated by the following recursive formula:

$$H_1 = [1]$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

HN= H2 $\otimes$ HN/2 …(2.4)

where $\otimes$ denotes the Kronecker product:

If $A = [aij]$ is an n1 by m1 matrix and $B = [bij]$ is an n2 by m2 matrix, then the kronecer product $A \otimes B$ is the matrix, [6]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1m1}B \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ a_{n11}B & a_{n12}B & . & . & . & a_{n1m1}B \end{bmatrix}$$ …(2.5)

The Main Result

Now we will give anther method to construct Selvester matrices which based on binary code to construct a Hadmard matrix. This method is shown in the follow step:

Step1: Consider the Hadamard matrix of order N as:

$$H_N = \begin{bmatrix} h_{00} & h_{01} & h_{02} & . & . & . & h_{om} \\ h_{10} & h_{11} & h_{12} & . & . & . & h_{1m} \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ h_{k0} & h_{K1} & h_{k2} & . & . & . & h_{km} \end{bmatrix} \quad \ldots(2.6)$$

where , hk,m $\forall k, m = 0,1,..., N-1$ are elements of Hadaamared matrix . These elements can be found as follow :

Step (2) : for k, m$\geq$ 0, we write the binary number of k,m as :

(k)b = (kn-1, kn-2, … k1, k0)2 = $\sum_{i=0}^{n-1} k_i 2^i$

$(m)_b = (m_{n-1}, m_{n-2}, \ldots, m_1, m_0)_2 = \sum_{i=0}^{n-1} m_i 2^i$

where
$$k_i \in \{0,1\}, \forall i = 0,1,..., n-1$$
$$m_i \in \{0,1\}, \forall i = 0,1,..., n-1$$
$$and\ n = \log_2^N$$

Step (3): put $h_{k,m} = (-1)^{\sum_{i=0}^{n-1} \oplus\, k_i m_i}$

where , $\oplus$ denotes addition modulo-2, (i-e- 0 $\oplus$ 0 = 0 , 0 $\oplus$ 1 = 1 $\oplus$ 0 = 1 and 1 $\oplus$ 1 = 0)

Illustrative Examples

Example (1): we know that, Hadamard matrix of order N=2 is given as, [6],:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

we will use our method to find each element in H2, as follow:

the Hadamard matrix of order N=2 is given by:

$$H_2 = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix}$$

all elements of Hadamard matrix of order $N = 2\ (i.e\ h_{0,0}, h_{0,1}, h_{1,0}, h_{1,1})$ can be found by using binary code as :

First , we write k,m= 0,1 as binary :

(0)b = (0)2
(1)b = (1)2

then , we apply $h_{k,m} = (-1)^{\sum_{i=0}^{n-1} \oplus\, k_i m_i}$ as follow :

$$h_{00} = h_{(0)(0)} = (-1)^{\sum_{i=0}^{0} \oplus\, k_i m_i} = (-1)^{k_0 m_0} = (-1)^{0*0} = (-1)^0 = 1$$
$$h_{01} = h_{(0)(1)} = (-1)^{\sum_{i=0}^{0} \oplus\, k_i m_i} = (-1)^{k_0 m_0} = (-1)^{0*1} = (-1)^0 = 1$$
$$h_{10} = h_{(1)(0)} = (-1)^{\sum_{i=0}^{0} \oplus\, k_i m_i} = (-1)^{k_0 m_0} = (-1)^{1*0} = (-1)^0 = 1$$
$$h_{11} = h_{(1)(1)} = (-1)^{\sum_{i=0}^{0} \oplus\, k_i m_i} = (-1)^{k_0 m_0} = (-1)^{1*1} = -1$$

From above , we have :

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example (2): Hadamand matrix of order N= 4 can be construct from Kroncker product,[6], as:

$$H_4 = H_2 \otimes H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

also, we can generate Hadamand matrix of order N= 4 by using binary code as:

the Hadamand matrix of order N= 4 is given by :

$$H_4 = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{bmatrix}$$

the element of Hadamard matrix of order N=22 = 4 can be found as :

first , we represent k,m as binary :

(0)b = (0,0)2
(1)b = (0,1)2
(2)b= (1,0)2
(3)b = (1,1)2

then

$$h_{00} = h_{(0,0)(0,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*0)\oplus(0*0)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{01} = h_{(0,0)(0,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*0)\oplus(0*1)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{02} = h_{(0,0)(1,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*1)\oplus(0*0)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{03} = h_{(0,0)(1,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*1)\oplus(0*1)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{10} = h_{(0,1)(0,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*0)\oplus(1*0)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{11} = h_{(0,1)(0,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*0)\oplus(1*1)} = (-1)^{0\oplus 1} = (-1)^1 = -1$$
$$h_{12} = h_{(0,1)(1,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*1)\oplus(1*0)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{13} = h_{(0,1)(1,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(0*1)\oplus(1*1)} = (-1)^{0\oplus 1} = (-1)^1 = -1$$
$$h_{20} = h_{(1,0)(0,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*0)\oplus(0*0)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{21} = h_{(1,0)(0,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*0)\oplus(0*1)} = (-1)^{0\oplus 0} = (-1)^0 = 1$$
$$h_{22} = h_{(1,0)(1,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*1)\oplus(0*0)} = (-1)^{1\oplus 0} = (-1)^1 = -1$$
$$h_{23} = h_{(1,0)(1,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*1)\oplus(0*1)} = (-1)^{1\oplus 0} = (-1)^1 = -1$$

$$h_{30} = h_{(1,1)(0,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*0)\oplus(1*0)} = (-1)^{0\oplus 0} = (-1)^{0} = 1$$

$$h_{31} = h_{(1,1)(0,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*0)\oplus(1*1)} = (-1)^{0\oplus 1} = (-1)^{1} = -1$$

$$h_{32} = h_{(1,1)(1,0)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*1)\oplus(1*0)} = (-1)^{1\oplus 0} = (-1)^{1} = -1$$

$$h_{33} = h_{(1,1)(1,1)} = (-1)^{\sum_{i=0}^{1} \oplus\, k_i m_i} = (-1)^{(1*1)\oplus(1*1)} = (-1)^{1\oplus 1} = (-1)^{0} = 1$$

then, we have :

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Example (3) : The Hadamard matrix of order N=23=8 , has the from :

$$H_8 = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} & h_{04} & h_{05} & h_{06} & h_{07} \\ h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} & h_{17} \\ h_{20} & h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} & h_{27} \\ h_{30} & h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} & h_{37} \\ h_{40} & h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} & h_{47} \\ h_{50} & h_{51} & h_{52} & h_{53} & h_{54} & h_{55} & h_{56} & h_{57} \\ h_{60} & h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} & h_{67} \\ h_{70} & h_{71} & h_{72} & h_{73} & h_{74} & h_{75} & h_{76} & h_{77} \end{bmatrix}$$

and the elements of Hadamard matrix of order N=23=8 can be found, see table (4.1).

also, we can generate Hadamand matrix of order N= 4 by using Kroncker product,[6] as:

$$H_8 = H_2 \otimes H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

From above, we see that our method is best and efficiency than Kroncker product.

Spectral Analysis

Spectral data are used in many applications in digital logic design. Some of them offer a possibility of function classification,[7], fault synthesis, signal processing, [8], and others. A Boolean function f(x1, x2,…, xn) can be transformed from the domain {0,1} in to the spectral domain by the linear transformation TY= Z, where T is a 2n× 2n orthogonal transform matrix, Y= (y0, y1,…,yn2-1)t is the two- valued truth vector of f(X1, x2,…, xn), and Z= (z0, z1, … zn2-1)t is the vector of spectral coefficients.

Hadamard matrices are used as transform matrices. Piotr porwik,[8], used selvester's product .In our work , we will use Hadamard matrices which based on binary code .

For example , let Y= (0,1,1,0,1,0,0,1)t be the truth vector of a given Boolean function, then

$$H_8 * Y = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} & h_{04} & h_{05} & h_{06} & h_{07} \\ h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} & h_{17} \\ h_{20} & h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} & h_{27} \\ h_{30} & h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} & h_{37} \\ h_{40} & h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} & h_{47} \\ h_{50} & h_{51} & h_{52} & h_{53} & h_{54} & h_{55} & h_{56} & h_{57} \\ h_{6,0} & h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} & h_{67} \\ h_{70} & h_{71} & h_{72} & h_{73} & h_{74} & h_{75} & h_{76} & h_{77} \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

from the table (4.1), we have

$$H_8 * Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

= (4,0,0,0,0,0,0,-4)t = (z0,z1,z2,z3,z4,z5,z6,z7)t

then z0= 4, z1= 0 , z2= 0 , z3= 0 , z4= 0 , z5= 0, z6= 0 , z7= - 4

Conclusion

1. In this paper , we give anther method to construct Sylvester Hadamard matrix of order N=2p from thrir elements . Which based on binary code .
2. Since , Hadamard matrices are orthogonal , take the value +1 and -1, and construction in binary code, they are can be used in communication system, image code , error correction code and others.
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Table (1) Hadamard matrix of order N=8 by using binary code.

$m=(m_{n-1},\ldots,m_1,m_0)$ $k=(k_{n-1},\ldots,k_1,k_0)$	$(0)_b=(0,0,0)_2$	$(1)_b=(0,0,1)_2$	$(2)_b=(0,1,0)_2$	$(3)_b=(0,1,1)_2$	$(4)_b=(1,0,0)_2$	$(5)_b=(1,0,1)_2$	$(6)_b=(1,1,0)_2$	$(7)_b=(1,1,1)_2$
$(0)_b=(0,0,0)_2$	$h_{(0,0,0),(0,0,0)}=1$	$h_{(0,0,0),(0,0,1)}=1$	$h_{(0,0,1),(0,1,0)}=1$	$h_{(0,0,0),(0,1,1)}=1$	$h_{(0,0,0),(1,0,0)}=1$	$h_{(0,0,0),(1,0,1)}=1$	$h_{(0,0,0),(1,1,0)}=1$	$h_{(0,0,0),(1,1,1)}=1$
$(1)_b=(0,0,1)_2$	$h_{(0,0,1),(0,0,0)}=1$	$h_{(0,0,1),(0,0,1)}=-1$	$h_{(0,0,0),(0,1,0)}=1$	$h_{(0,0,0),(0,1,1)}=-1$	$h_{(0,0,0),(1,0,0)}=1$	$h_{(0,0,0),(1,0,1)}=-1$	$h_{(0,0,0),(1,1,0)}=1$	$h_{(0,0,0),(1,1,1)}=-1$
$(2)_b=(0,1,0)_2$	$h_{(0,0,0),(0,0,0)}=1$	$h_{(0,0,0),(0,0,1)}=1$	$h_{(0,0,1),(0,1,0)}=-1$	$h_{(0,0,1),(0,1,1)}=-1$	$h_{(0,0,1),(1,0,0)}=1$	$h_{(0,0,1),(1,0,1)}=1$	$h_{(0,0,0),(1,1,0)}=-1$	$h_{(0,0,0),(1,1,1)}=-1$
$(3)_b=(0,1,1)_2$	$h_{(0,1,1),(0,0,0)}=1$	$h_{(0,1,1),(0,0,1)}=-1$	$h_{(0,1,1),(0,1,0)}=-1$	$h_{(0,1,1),(0,1,1)}=1$	$h_{(0,1,1),(1,0,0)}=1$	$h_{(0,1,1),(1,0,1)}=-1$	$h_{(0,0,0),(1,1,0)}=-1$	$h_{(0,0,0),(1,1,1)}=1$
$(4)_b=(1,0,0)_2$	$h_{(1,0,0),(0,0,0)}=1$	$h_{(1,0,0),(0,0,1)}=1$	$h_{(1,0,1),(0,1,0)}=1$	$h_{(1,0,0),(0,1,1)}=1$	$h_{(1,0,0),(1,0,0)}=-1$	$h_{(1,0,0),(1,0,1)}=-1$	$h_{(1,0,0),(1,1,0)}=-1$	$h_{(1,0,0),(1,1,1)}=-1$
$(5)_b=(1,0,1)_2$	$h_{(1,0,1),(0,0,0)}=1$	$h_{(1,0,1),(0,0,1)}=-1$	$h_{(1,0,1),(0,1,0)}=1$	$h_{(1,0,1),(0,1,1)}=-1$	$h_{(1,0,1),(1,0,0)}=-1$	$h_{(1,0,1),(1,0,1)}=1$	$h_{(1,0,1),(1,1,0)}=-1$	$h_{(1,0,1),(1,1,1)}=1$
$(6)_b=(1,1,0)_2$	$h_{(1,1,0),(0,0,0)}=1$	$h_{(1,1,0),(0,0,1)}=1$	$h_{(1,1,0),(0,1,0)}=-1$	$h_{(1,1,0),(0,1,1)}=-1$	$h_{(1,1,0),(1,0,0)}=-1$	$h_{(1,1,0),(1,0,1)}=-1$	$h_{(1,1,0),(1,1,0)}=1$	$h_{(1,1,0),(1,1,1)}=1$
$(7)_b=(1,1,1)_2$	$h_{(1,1,1),(0,0,0)}=1$	$h_{(1,1,1),(0,0,1)}=-1$	$h_{(0,0,1),(0,1,0)}=-1$	$h_{(1,1,1),(0,1,1)}=1$	$h_{(1,1,1),(1,0,0)}=-1$	$h_{(1,1,1),(1,0,1)}=1$	$h_{(1,1,1),(1,1,0)}=1$	$h_{(1,1,1),(1,1,1)}=-1$

إنشاء مصفوفات سلفستر - هادمارد (Sylvester-Hadamard Matrices) باستخدام التشفير الثنائي

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الخلاصة

طريقة بسيطة قدمت لتعريف مصفوفات سلفستر هادمارد (Sylvester-Hadamard Matrices) بلغة التشفير الثنائي . بنيـت هذه الطريقة على تمثيل الاعداد الطبيعية كتشفير ثنائي الذي ياخذ قيمتين فقط 1 او 0 . حيث مصفوفات هادمارد المولدة استخدمت لايجاد معاملات الطيف لدالة بولين (Boolean function).