Effect of General Non-Slip Condition and MHD on Accelerated Flows of a Viscoelastic Fluid With the Fractional Burgers' Model

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Abstract

In this paper, the effect of general non-slip condition and magnetohydrodynamic (MHD) on accelerated flows of a viscoelastic fluid with the fractional Burgers’ model are studied. The velocity field of the flow is described by a fractional partial differential equation. By using Fourier sine transform and Laplace transform, exact solutions for the velocity distribution and shear stress are obtained for flow induced by variable accelerated plate. These solutions, are presented under integral and series forms in terms of the generalized Mittag-Leffler function, as the sum of two terms. The first term represents the velocity field corresponding to a Newtonian fluid, and the second term gives the non-Newtonian contributions to the general solutions. The similar solutions for second grad, Maxwell and Oldroyd-B fluids with fractional derivatives as well as those for the ordinary models are obtained as the limiting cases of our solutions. Moreover, in the special cases when $\alpha = \beta = 1$, as it was to be expected, our solutions tend to the similar solutions for an ordinary Burgers’ fluid. While the MATHEMATICA package is used to draw the figures velocity components in the plane.

Keywords: {Non-Slip Condition, Magnetohydrodynamic (MHD), Oldroyd-B Fluid, Fractional Burgers' Model }

تأثيــر شرط عدم الانزلاق العام و المجال المغناطيسي الهيدرودينامـيكـي على التدفقات المتسارعة للموائع اللزجة المرنة مـــع نمــوذج بيركـــر للمشتقات الكسرية

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المستخلص

في هذا البحث تم دراسة تأثير شرط عدم الانزلاق العام و المجـال المغناطيســي الهيدروديناميكـي علـى التدفقات المتسارعة للموائع اللزجة مع نموذج "بيركر". حقل سرعة التدفق وصف بواسطة معادلة تفاضلية جزئية كسرية. باستخدام تحويـلات كل مـن فوريـر و لابـــلاس، تم الحصول على حلول اجهاد القص والحلول الدقيقة لتوزيع سرعة التدفق الناجم عن لوحة التسريع المتغيرة. هذه الحـلول، قدمت بصيغـة التكامل والمتسلسلات بدلالة دالة ميتاج لفلر العامة، كما انها تظهر بصيغة جمع لمصطلحين. المصطلح الاول يمثل حقل السرعة للموائع النيوتونية لأداء الحركة نفسها، والمصطلح الثاني يمثل حقل سرعة المائع اللانيوتوني . تم الحصول على حلول مماثلة لموائع من الرتبة الثانية مثل ماكسويل، و اولدرويد من النمط بي.ذات مشتقات كسرية، وكذلك كحالات محددة تم الحصول على النماذج الاعتيادية. بالأضافة الى ذلك، وكحالات خاصة، تم تغطيتها، عندما $\alpha = \beta = 1$ كمــا كان متوقعا، حلولنا تميل الى حلـول مماثلة لموائع بيركــر الاعتيادي. بينما برنامج الماثيماتيكا أستعمل لرسم اشكال مكونات السرعة في المستوي.

الكلمات المفتاحية:
{ شرط عدم الانزلاق ، المجال الغناطيسي الهيدروديناميكي ، مائع اولدرايد من النمط بي ، نموذج بيركر الكسري}

1. Introduction

Understanding non- Newtonian fluid flows behavior becomes increasingly important as the application of non-Newtonian fluids perpetuates through various industries, including polymer processing and electronic packaging , paints , oils liquid polymers, glycerin , chemical , geophysics and biorheology. However, there is no model which can alone predict the behaviors of all non-Newtonian fluids. Amongst the existing model, rate type models have special importance and many researchers are using equations of motion of Maxwell and Oldroyd-fluid flows[1], [2], [3]. Recently, a thermodynamic framework has been put into place to develop a rate type model known as Burgers' model which is used to describe the motion of the earth's mantle. This model is mostly used to model other geological structures, such as Olivine rocks [4]. M. Khan, S. Hyder Ali, Haitao Qi. (2007)[5] constructed the exact solutions for the accelerated flows of a generalized Oldroyd-B fluid. The fractional calculus approach is used in the constitutive relationship of a viscoelastic fluid. The velocity field and the adequate tangential stress that is induced by the flow due to constantly accelerating plate and flow due to variable accelerating plate are determined by means of discrete Laplace transform. C. Fetecau,T. Hayat and M.Khan. (2008)[6] concerned with the study of unsteady flow of an Oldroyed-B fluid produced by a suddenly moved plane wall between two side walls perpendicular to the plane are established by means of the Fourier sine transforms. M. Khan, S. Huder Ali, Haitao Qi. (2009)[7] Studied the accelerated flows for a viscoelastic fluid governed by the fractional Burgers’ model. The velocity field of the flow is described by a fractional partial differential equation. Liancun Zheng, Yaqing Liu, Xinxin Zhang. (2011)[8] research for the magnetohydrodynamic (MHD) flow of an incompressible generalized Oldroyd-B fluid due to an infinite accelerating plate. Mahmood, Hind Shaker, (2012)[9], studied the effect of MHD on Accelerated Flows of a Viscoelastic Fluid With the Fractional Burgers' Model. Thesis submitted to university of Baghdad. The motion of the fluid is produced by the accelerated plate, which at time $t = 0^+$ begins to slide in its plane with a velocity At . The solutions are established by means of Fourier sine and Laplace transforms. This paper is organized as follows, the basic equations and the solution for variable accelerated flow are given in section 2. Section 3 contains the results and discussion. The paper ends with draw the figures of velocity component in the plane.

2. Governing Equations:

The constitutive equations for an incompressible fractional Burgers’ fluid are given by [7]

$$T = -PI + S, \qquad (1+\lambda_1^{\alpha}\tilde{D}_t^{\alpha} + \lambda_2^{\alpha}\tilde{D}_t^{2\alpha})S = \mu(1+\lambda_3^{\beta}\tilde{D}_t^{\beta})A \qquad \ldots(1)$$

Where T is the Cauchy stress tensor,-PI denotes the indeterminate spherical stress, S the extra stress tensor, $A = L + L^T$ the first Rivlin-Ericksen tensor, where L the velocity gradient, μ the dynamic viscosity of the fluid, λ_1 and $\lambda_3 (< \lambda_1)$ the relaxation and retardation times, respectively, λ_2 is the new material parameter of the Burgers’ fluid, α and β the fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$ and $\tilde{D}_t^{p}$ the upper connected fractional derivative defined by

$$\tilde{D}_t^P S = D_t^P S + v \cdot \nabla S - LS - SL^T, \qquad \ldots(2)$$

In which $D_t^p (= \partial_t^p)$ is the fractional differentiation operator of order p with respect to t and is defined as [10]

$$D_t^p [f(t)] = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^p} d\tau , \qquad 0 \le p \le 1 \qquad \ldots(3)$$

Here $\Gamma(.)$ denotes the Gamma function and

$$\tilde{D}_t^{2P} S = \tilde{D}_t^P (\tilde{D}_t^P S), \qquad \ldots(4)$$

The equations of motion in absence of body force can be described as

$$\rho \frac{d\vec{v}}{dt} = \nabla \cdot \vec{T}, \qquad \ldots(5)$$

Where ρ is the density of the fluid and *d/dt* represents the material time derivative.
Since the fluid is incompressible, it can undergo only is isochoric motion and hence

$$\nabla \cdot \vec{v} = 0, \qquad \ldots(6)$$

For the following problem of unidirectional flow, the intrinsic velocity field takes the form

$$\vec{v} = [u(y,t), 0, 0] \qquad \ldots(7)$$

Where $u(y,t)$ is the velocity in the x-coordinates direction. For this velocity field, the constraint of incompressibility (6) is automatically satisfied, we also assume that the extra stress S depends on y and t only. Substituting Eq. (7) into (1), (5) and taking account of the initial conditions $S(y,0) = \partial_t S(y,0) = 0, \quad y > 0$. (i.e. the fluid being at rest up to the time t = 0). For the components of the stress field S, we have $S_{yy} = S_{zz} = S_{xz} = S_{yz} = 0$ and $S_{xy} = S_{yx}$, this yields

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \qquad \ldots(8)$$

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) S_{xy} = \mu (1 + \lambda_3^\beta D_t^\beta) \frac{\partial u}{\partial y} \qquad \ldots(9)$$

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) S_{xx} - 2 S_{xy} [\lambda_1^\alpha + \lambda_2^\alpha D_t^\alpha] \frac{\partial u}{\partial y} - 2\lambda_2^\alpha \frac{\partial u}{\partial y} D_t^\alpha S_{xy} = -2\mu \lambda_3^\beta \left(\frac{\partial u}{\partial y} \right)^2 \qquad \ldots(10)$$

Consider that the conducting fluid is permeated by an imposed magnetic field B_0 which acts in the positive y-direction. In the low-magnetic Reynolds number approximation, the magnetic body force is representation by $\sigma B_0 u$. Consider an incompressible fractional Burgers' fluid lying over an infinitely extended plate which is situated in the (x,z) plane.

Initially, the fluid is at rest and at time $t = 0^{+}$, the infinite plate to slide in its own plane with a motion of the variable acceleration A. Owing to the shear, the fluid above the plate is gradually moved. Under these considerations, in the absence of a pressure gradient in the x-direction, the equation of motion (8) yields the following scalar equations:

$$\rho \frac{\partial u}{\partial t} = \frac{\partial S_{xy}}{\partial y} - \sigma B_0^2 u \qquad \dots(11)$$

where ρ is the constant density of the fluid.

And eliminating S_{xy} between Eqs. (9) and (11) ,we arrive at the following governing fractional differential equation

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\frac{\partial u}{\partial t} = \upsilon(1+\lambda_3^{\beta} D_t^{\beta})\frac{\partial^2 u}{\partial y^2} - M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})u \qquad \dots(12)$$

where $\upsilon = \frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M = \frac{\sigma B_{\circ}^2}{\rho}$.

The associated initial and boundary condition are follows:

Initial condition:

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = \frac{\partial^2 u(y,0)}{\partial t^2} = 0\ , \quad y > 0 \qquad \dots(13)$$

Boundary conditions:

$$u(0,t) = At^{b} \quad , \quad t > 0 \quad , \qquad \dots(14)$$

Moreover, the natural conditions are

$$u(y,t)\ ,\frac{\partial u(y,t)}{\partial y} \to 0 \ \ as \ \ y \to \infty \ \ and \ \ t > 0 \qquad \dots(15)$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

Now, apply Fourier sine transform[11] to Eq. (12) and take into account the boundary conditions (14) and (15), yields

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\frac{\partial U_S(\xi,t)}{\partial t} = \upsilon(1+\lambda_3^{\beta} D_t^{\beta})\left(\sqrt{\frac{2}{\pi}}\ \xi A t^{b} - \xi^2 U_s(\xi,t)\right)$$
$$- M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\, U_s(\xi,t) \qquad \dots(16)$$

where the Fourier sine transform $U_s(\xi,t)\ of\ u(y,t)$ has to satisfy the conditions

$$U_s(\xi,0) = \frac{\partial U_s(\xi,0)}{\partial t} = \frac{\partial^2 U_s(\xi,0)}{\partial t^2} = 0\ ; \quad \xi > 0. \qquad \dots(17)$$

Let $\overline{U}_s(\xi,s)$ be the Laplace transform of $U_s(\xi,t)$ defined by

$$\overline{U}_s(\xi,s) = \int_0^{\infty} U_s(\xi,t)\exp(-st)\,dt \quad , \ s > 0. \qquad \dots(18)$$

Taking the Laplace transform of Eq.(16), having in mind the initial conditions (17), we get[10]

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}A\frac{b!}{s^{b+1}}\left[\frac{\upsilon\,\xi(1+\lambda_3^{\beta}s^{\beta})}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\upsilon\xi^2+\upsilon\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}\right] \quad ...(19)$$

In order to obtain $U_s(\xi,t)=L^{-1}\{\overline{U}_s(\xi,s)\}$ with L^{-1} as the inverse Laplace transform operator and to avoid the lengthy procedure of residues and contour integral, we apply the discrete Laplace transform method. However, for a more suitable presentation of the final results, Eq. (19) is rewritten in the equivalent form

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}A\frac{b!}{s^{b+1}}\frac{1}{\xi}\left[1-\frac{s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha}}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\upsilon\xi^2+\upsilon\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}\right] \quad ...(20)$$

And then

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}A\frac{b!}{s^{b+1}}\frac{1}{\xi}-\sqrt{\frac{2}{\pi}}A\frac{1}{\xi}\left[\frac{s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha}}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\upsilon\xi^2+\upsilon\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}\right]\frac{b!}{s^{b+1}} \quad ...(21)$$

Hence, Eq. (21) can be written under the form of a double series as (using $\frac{1}{z+a}=\sum_{k=0}^{\infty}(-1)^k\frac{z^k}{a^{k+1}}$ and $(1+b)^k=\sum_{m=0}^{k}\frac{k!b^m}{m!(k-m)!}$)

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\,A\frac{b!}{s^{b+1}}\frac{1}{\xi}-\sqrt{\frac{2}{\pi}}\,A\frac{1}{\xi}[(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})$$

$$\sum_{m=0}^{\infty}(-1)^m\frac{\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}}{(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M))^{m+1}}$$

$$\lambda_1^{\alpha(-m+i-d-1)}\,\lambda_2^{\alpha(l-i)}\,\lambda_3^{\beta d}\,M^{j-d}\,(\upsilon\xi^2)^d\,m!\,s^{\delta}]\frac{b!}{s^{b+1}} \quad ...(22)$$

In which $\delta=m+2\alpha l-j-\alpha i+\beta d-\alpha d.$

Now, applying the inversion formula term by term for the Laplace transform, Eq.(22) yields

$$U_s(\xi,t)=\sqrt{\frac{2}{\pi}}\,\frac{1}{\xi}At^b-\sqrt{\frac{2}{\pi}}\,\frac{A}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}$$

$$\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\quad \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d$$

$$\times[t^{(\alpha+1)m+(\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+\lambda_1^{\alpha}\ t^{(\alpha+1)m-\delta-1}\ E^{(m)}_{(\alpha+1),-\delta}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\,t^{\alpha+1})$$

$$+\lambda_2^{\alpha}\ t^{(\alpha+1)m+(-\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\ t^{(\alpha+1)m+(\alpha+1-\delta)-1}\ E^{(m)}_{(\alpha+1),(\alpha+1-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\lambda_1^{\alpha}t^{(\alpha+1)m+(1-\delta)-1}\ E^{(m)}_{(\alpha+1),(1-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\lambda_2^{\alpha}t^{(\alpha+1)m+(1-\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(1-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})]*t^b \qquad \ldots(23)$$

where "$*$" represents the convolution of two functions and

$$E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)},\quad \lambda,\mu>0, \qquad \ldots(24)$$

Denotes the generalized Mittag-Leffler function with

$$E^{(k)}_{\lambda,\mu}(z)=\frac{d^k}{dz^k}\ E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!\,z^n}{n!\Gamma(\lambda n+\lambda k+\mu)}\ . \qquad \ldots(25)$$

Here, we used the following property of the generalized Mittad-Leffler function[11]

$$L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\}=t^{\lambda k+\mu-1}E^{(k)}_{\lambda,\mu}\ (\pm ct^{\lambda}),\quad (\mathrm{Re}(s)>|c|^{1/\lambda}). \qquad \ldots(26)$$

and

$$L^{-1}(F(s)G(s))=f(t)*g(t) \qquad \ldots(27)$$

Finally, inverting (23) by the Fourier transform[11] we find for the velocity $u(y,t)$ the expression

$$u(y,t) = At^b - \frac{2}{\pi} A \int_0^\infty \int_0^t \frac{\sin(\xi y)}{\xi} \sum_{m=0}^{\infty} (-1)^m \sum_{l=0}^{m} \frac{1}{l!(m-l)!} \sum_{j=0}^{l} \frac{l!}{j!(l-j)!}$$

$$\sum_{i=0}^{j} \frac{j!}{i!(j-i)!} \sum_{d=0}^{i} \frac{i!}{d!(i-d)!} \lambda^{\alpha(-m+i-d-1)} \lambda_2^{\alpha(l-i)} \lambda_3^{\beta d} M^{j-d} (\upsilon \xi^2)^d$$

$$\times[\tau^{(\alpha+1)m+(\alpha-\delta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\delta))}$$

$$+\lambda_1^\alpha \tau^{(\alpha+1)m-\delta-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\delta))}$$

$$+\lambda_2^\alpha \tau^{(\alpha+1)m+(-\alpha-\delta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\delta))}$$

$$+M \tau^{(\alpha+1)m+(\alpha+1-\delta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\delta))}$$

$$+M \lambda_1^\alpha \tau^{(\alpha+1)m+(1-\delta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\delta))}$$

$$+M \lambda_2^\alpha \tau^{(\alpha+1)m+(1-\alpha-\delta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\delta))}](t-\tau)^b d\tau d\xi \quad ...(28)$$

where

$$f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau \ \ and \ \ \int_0^\infty \frac{\sin(\xi y)}{\xi} d\xi = \frac{\pi}{2} \quad ...(29)$$

To obtain the expression for the shear stress $S_{xy}(y,t)$ we first apply the Laplace transform to Eq. (9) and using the initial condition $S(y,0) = \frac{\partial S(y,0)}{\partial t} = 0$, yields:

$$\bar{S}_{xy}(y,s) = \mu \frac{(1+\lambda_3^\beta s^\beta)}{1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha}} \frac{\partial \bar{U}(y,s)}{\partial y} \quad ...(30)$$

Take the part $\frac{1}{1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha}}$ and using $\frac{1}{1+\gamma} = \sum_{i=0}^{\infty} (-1)^i \gamma^i$ and $(1+b)^k = \sum_{m=0}^{k} \frac{k!b^m}{m!(k-m)!}$, then

$$\frac{1}{1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha}}=\sum_{r=0}^{\infty}(-1)^r(\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})^r$$

$$=\sum_{r=0}^{\infty}(-1)^r(\lambda_1^\alpha s^\alpha)^r\left(1+\frac{\lambda_2^\alpha}{\lambda_1^\alpha}s^\alpha\right)^r$$

$$=\sum_{r=0}^{\infty}(-1)^r(\lambda_1^\alpha s^\alpha)^r\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\left(\frac{\lambda_2^\alpha}{\lambda_1^\alpha}s^\alpha\right)^p \qquad \text{... (31)}$$

substituting Eq. (31) into Eq. (30) to get

$$\bar{S}_{xy}(y,s)=\mu\sum_{r=0}^{\infty}(-1)^r\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\lambda_1^{\alpha(r-p)}\lambda_2^{\alpha p}s^{\alpha(r+p)}(1+\lambda_3^\beta s^\beta)\frac{\partial\bar{U}(y,s)}{\partial y} \qquad \text{... (32)}$$

The image function $\bar{U}(y,s)$ can immediately be obtained through Eq. (22) . Consequently, evaluating $\frac{\partial\bar{U}(y,s)}{\partial y}$ from the mentioned equation and introducing it into Eq. (32), yields

$$\bar{S}_{xy}(y,s)=\frac{\partial}{\partial y}\Big[\sqrt{\frac{2}{\pi}}\mu\frac{A}{\xi}\sum_{r=0}^{\infty}(-1)^r\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\lambda_1^{\alpha(r-p)}\lambda_2^{\alpha p}s^{\alpha(r+p)}(1+\lambda_3^\beta s^\beta)\frac{b!}{s^{b+1}}$$
$$-\sqrt{\frac{2}{\pi}}\mu\frac{A}{\xi}\Big[\big((s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha}$$
$$+\lambda_3^\beta(s^{\beta+1}+\lambda_1^\alpha s^{\beta+\alpha+1}+\lambda_2^\alpha s^{\beta+2\alpha+1}+Ms^\beta+M\lambda_1^\alpha s^{\beta+\alpha}+M\lambda_2^{2\alpha}s^{\beta+2\alpha})\big)$$
$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+r}\frac{\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}}{\left(s^{\alpha+1}+\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M)\right)^{m+1}}$$
$$\lambda_1^{\alpha(r-p-m+i-d-1)}\lambda_2^{\alpha(p+l-i)}\lambda_3^{\beta d}M^{j-d}(\upsilon\xi^2)^d\, m!\,s^\eta\Big]\frac{b!}{s^{b+1}}\Big] \qquad \text{... (33)}$$

in which $\eta=\delta+\alpha(r+p).$

Now, applying the inversion formula term by term for the Laplace transform Here, we used the following property of the generalized Mittad-Leffler function[11]

$$L^{-1}\left\{\frac{k!s^{\lambda-\mu}}{(s^\lambda\mp c)^{k+1}}\right\}=t^{\lambda k+\mu-1}E_{\lambda,\mu}^{(k)}(\pm ct^\lambda),\quad (\mathrm{Re}(s)>|c|^{1/\lambda}). \qquad \text{...(34)}$$

then Eq.(33) yields

$$S_{xy}(y,t)=\frac{\partial}{\partial y}[\sqrt{\frac{2}{\pi}}\frac{A}{\xi}\mu\sum_{r=0}^{\infty}(-1)^r\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\lambda_1^{\alpha(r-p)}\lambda_2^{\alpha p}$$
$$(\frac{t^{-1-(r+p)\alpha}}{\Gamma(-(r+p)\alpha)}+\lambda_3^{\beta}\frac{t^{-1-(r+p)\alpha-\beta}}{\Gamma(-(r+p)\alpha-\beta)})*t^b$$
$$-\sqrt{\frac{2}{\pi}}\mu\frac{A}{\xi}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+r}\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}$$
$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}$$
$$.\lambda_1^{\alpha(r-p-m+i-d-1)}\ \lambda_2^{\alpha(p+l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d$$
$$\times[[t^{(\alpha+1)m+(\alpha-\eta)-1}\ E_{(\alpha+1),(\alpha-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+\lambda_1^{\alpha}\ t^{(\alpha+1)m-\eta-1}\ E_{(\alpha+1),-\eta}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\ t^{\alpha+1})$$
$$+\lambda_2^{\alpha}\ t^{(\alpha+1)m+(-\alpha-\eta)-1}\ E_{(\alpha+1),(-\alpha-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\ t^{(\alpha+1)m+(\alpha+1-\eta)-1}\ E_{(\alpha+1),(\alpha+1-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_1^{\alpha}t^{(\alpha+1)m+(1-\eta)-1}\ E_{(\alpha+1),(1-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_2^{\alpha}t^{(\alpha+1)m+(1-\alpha-\delta)-1}\ E_{(\alpha+1),(1-\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})]$$
$$+\lambda_3^{\beta}[t^{(\alpha+1)m+(\alpha-\beta-\eta)-1}\ E_{(\alpha+1),(\alpha-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+\lambda_1^{\alpha}\ t^{(\alpha+1)m+(-\beta-\eta)-1}\ E_{(\alpha+1),(-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\ t^{\alpha+1})$$
$$+\lambda_2^{\alpha}\ t^{(\alpha+1)m+(-\alpha-\beta-\eta)-1}\ E_{(\alpha+1),(-\alpha-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\ t^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1}\ E_{(\alpha+1),(\alpha+1-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_1^{\alpha}t^{(\alpha+1)m+(1-\beta-\eta)-1}\ E_{(\alpha+1),(1-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_2^{\alpha}t^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1}\ E_{(\alpha+1),(1-\alpha-\beta-\eta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})]]*t^b] \qquad ...(35)$$

where "$*$" represents the convolution of two functions. Finally, inverting (35) by the Fourier transform, and using $f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$ and denotes the generalized Mittag-Leffler function with

$$E_{\lambda,\mu}^{(k)}(z) = \frac{d^k}{dz^k} E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!\, z^n}{n!\,\Gamma(\lambda n + \lambda k + \mu)} . \qquad \ldots(36)$$

then we have the shear stress $S_{xy}(y,t)$ in the expression

$$\begin{aligned}
S_{xy}(y,t) = & -\frac{2}{\pi} A\mu \int_0^{\infty}\int_0^{2} \cos(\xi y) \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \sum_{l=0}^{m} \frac{1}{l!(m-l)!} \sum_{p=0}^{i} \frac{r!}{p!(r-p)!} \\
& \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} \sum_{d=0}^{i} \frac{i!}{d!(i-d)!} \\
& .\lambda^{\alpha(r-p-m+i-d-1)} \lambda_2^{\alpha(p+l-i)} \lambda_3^{\beta d} M^{j-d} (\upsilon \xi^2)^d \\
& \times[[\tau^{(\alpha+1)m+(\alpha-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\eta))} \\
& + \lambda_1^{\alpha} \tau^{(\alpha+1)m-\eta-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\eta))} \\
& + \lambda_2^{\alpha} \tau^{(\alpha+1)m+(-\alpha-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\eta))} \\
& + M\tau^{(\alpha+1)m+(\alpha+1-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\eta))} \\
& + M\lambda_1^{\alpha} \tau^{(\alpha+1)m+(1-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\eta))} \\
& + M\lambda_2^{\alpha} \tau^{(\alpha+1)m+(1-\alpha-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\eta))}] \\
& + \lambda_3^{\beta}[\tau^{(\alpha+1)m+(\alpha-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\beta-\eta))} \\
& + \lambda_1^{\alpha} \tau^{(\alpha+1)m+(-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\beta-\eta))} \\
& + \lambda_2^{\alpha} \tau^{(\alpha+1)m+(-\alpha-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\beta-\eta))} \\
& + M\tau^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\beta-\eta))} \\
& + M\lambda_1^{\alpha} \tau^{(\alpha+1)m+(1-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\beta-\eta))} \\
& + M\lambda_2^{\alpha} \tau^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1} \sum_{n=0}^{\infty} \frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\beta-\eta))}]] \\
& .(t-\tau)^b \, d\tau \, d\xi \qquad \ldots (37)
\end{aligned}$$

3. Results and Discussion:

This section includes the discussion relevant to the velocity profiles for variable accelerated flow. Comparison are preformed between ordinary and fractional models for both the fluids (Oldroyd-B and Burgers' fluids) for various values of $\alpha, \beta, \lambda_2, and\ t$. The effect of non-slip condition b and magnetic body force M are also discussion. We interpret these results with respect to the variation of emerging parameter of interest. For the sake of simplicity, we take $A=1$, $\upsilon = 0.002$ in all figures.

Figs.(1,2) show the profiles of the velocity with two fixed different values of t for different values of fractional parameters α and β of the models. From these figures, it obvious to note that the order of the fractional parameter α have strong effect on the velocity profiles. It is noted that as α increases there is increasing in the velocity value in the case of Burgers' fluid. However, α has opposite influence in case of Oldroyd-B fluid. Whereas β shows small effect on the velocity profiles. It is note that as β increases there is very small increasing in the velocity value in both the fluids (Oldroyd-B and Burgers' fluids).

The features of the rheological parameter of the Burgers' fluid can be observed from Fig.(3). From this figure, it appears that the velocity is strong function of the rheological parameter λ_2 of the Burgers' fluid. It can be seen that the velocity is a increasing function of the rheological parameter λ_2 of the Burgers' fluid. However, this result cannot be generalized for other chosen values of λ_2 since the behavior of λ_2 is non-monotonous. Furthermore, it is interesting to observe that the effect of rheology of the fluid on the flow is much stronger in fractional models than those for the ordinary models.

Fig.(4) illustrate the variation of the velocity profiles for different values of t. It is obvious to note that the velocity is an increasing function of time for both models. A comparison shows that the velocity profiles for fractional Burgers' fluid are much greater in magnitude than those of ordinary Burgers' fluid.

Fig.(5) shows the velocity changes with the magnetic field parameter. It is observed that the velocity decrease as the magnetic field M parameter increasing. Fig.(6) shows the velocity changes with the non-slip condition. It is observed that the velocity will increase as the power of b (non-slip condition) increasing.

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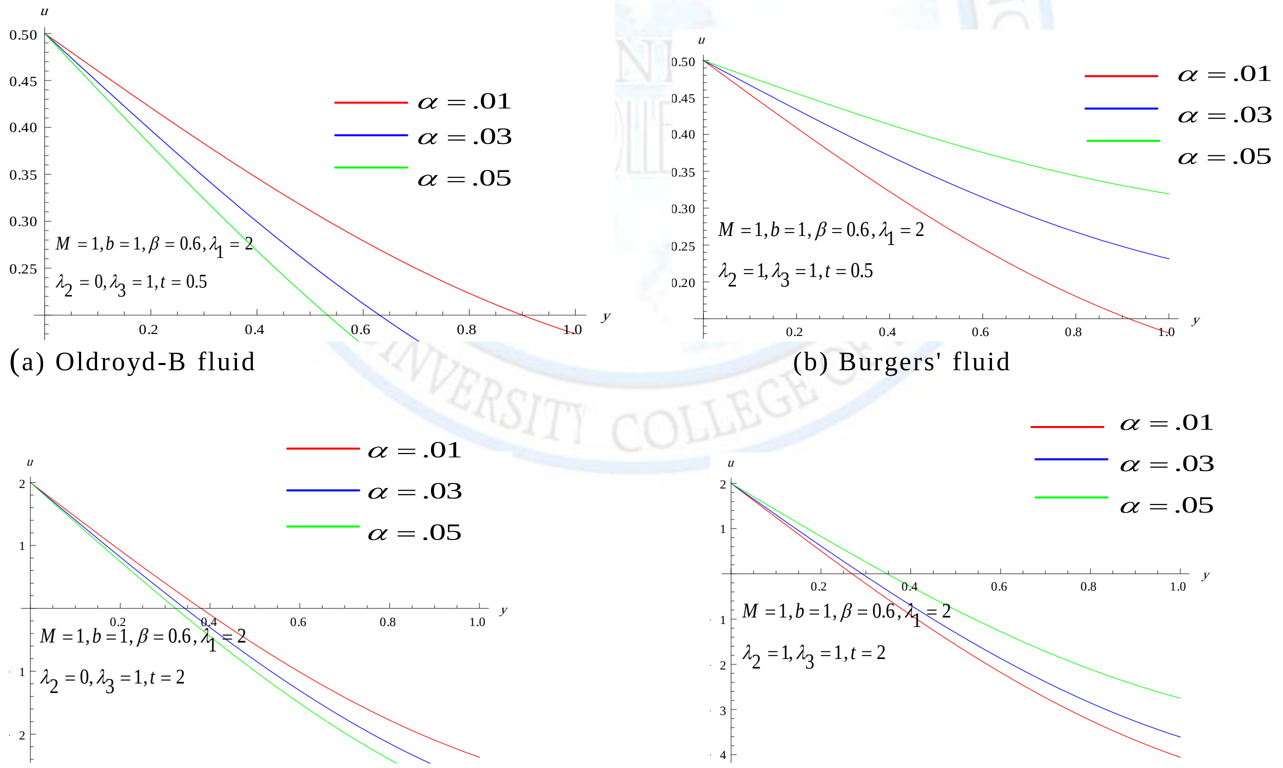
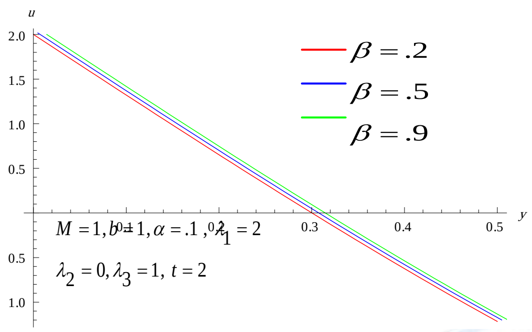
Fig.(1) Velocity u(y,t) versus y for different value of α when the other parameters are fixed.



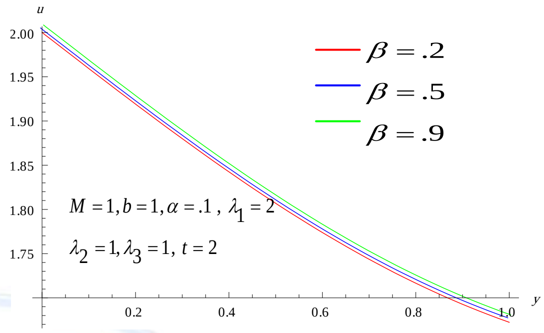
(a) Oldroyd-B fluid (b) Burgers' fluid

(c) Oldroyd-B fluid (d) Burgers' fluid

Fig.(2) Velocity u(y,t) versus y for different value of β when the other parameters are fixed.

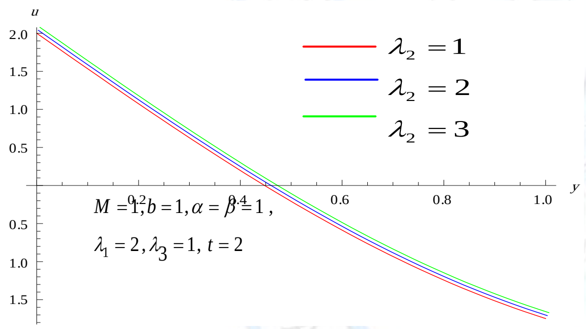


(a) Oldroyd-B fluid

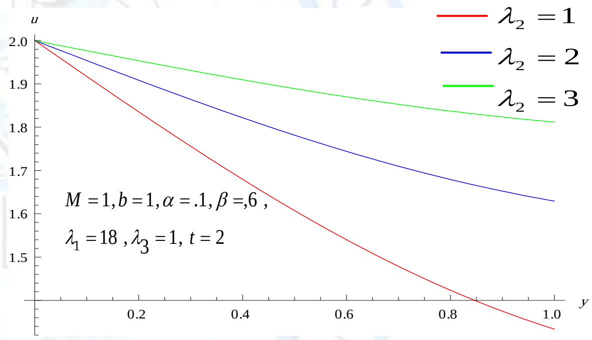


(b) Burgers' fluid

Fig.(3) Velocity u(y,t) versus y for different value of λ_2 when the other parameters are fixed.

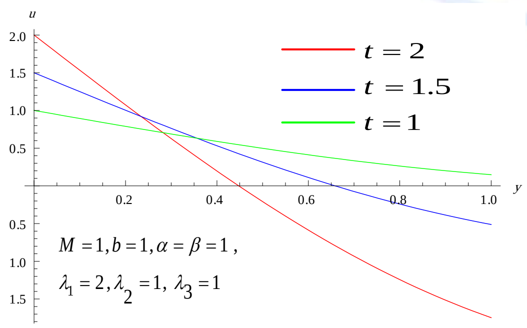


(a) Ordinary model

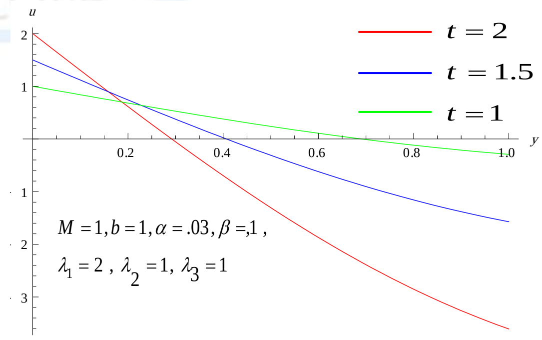


(b) Fractional model

Fig.(4) Velocity u(y,t) versus y for different value of t when the other parameters are fixed.

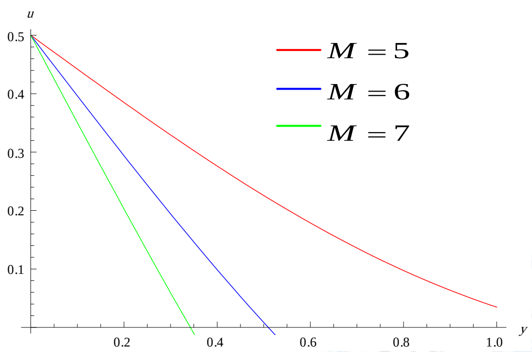


(a) Ordinary model



(b) Fractional model

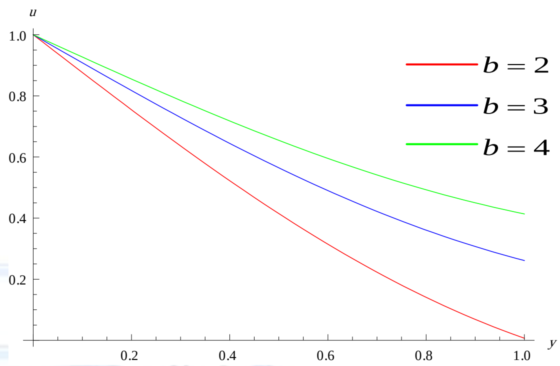
Fig.(5) Velocity u(y,t) versus y for different value of M when the other parameters are fixed.



$b = 1, \alpha = .1, \beta =,6$,

$\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1, t = .5$

Fig.(6) Velocity u(y,t) versus y for different value of b when the other parameters are fixed.



$M = 1, \alpha = .01, \beta =,6$,

$\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1, t = 1$