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ON DIFFERENCE CORDIAL GRAPHS

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Abstract

In this paper we introduce some results in difference cordial graphs and the difference cordial labeling for some families of graphs as: ladder,triangular ladder,grid,step ladder and two sided step ladder graph. Also we discussed some families of graphs which may be difference cordial or not,such as diagonal ladder and some types of one-point union graphs.

Mathematics Subject Classification: 05C78

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1 Introduction

In this paper we will deal with finite,simple and undirected graphs. By the expression $G = (V, E)$ we mean a simple undirected graph with vertex set V, $|V|$ is called the order of graph and edge set E, $|E|$ is called its size.

Graph labeling connects many branches of mathematics and is considered one of important blocks of graph theory, for more details see [3]. Cordial labeling was first introduced in 1987 by Cahit [1], then there was a major effort in this area made this topic growing steadily and widely,see[2].

In [4] Ponraj ,Shathish Naraynan and Kala introduce the notions of difference cordial labeling for finite undirected and simple graph,as in the following definition :

Definition 1.1 *Let $G = (V, E)$ be a (p, q) graph,and f be a map from $V(G)$ to $1, 2, ..., p$. For each edge uv assign the label $|f(v) - f(u)|$, f is called a difference cordial labeling if f is one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ denotes the number of edges labeled with 1 while $e_f(0)$ denotes the number of edges not labeled with 1. A graph with a difference cordial labeling is called a difference cordial graph [4].*

Ponraj et al. show every graph is a subgraph of a difference cordial graph and any $r - regular$ graph with $r \geq 4$ is not difference cordial graph,every path and cycle are difference cordial graphs,the star graph $K_{1,n}$ is difference cordial if and only if $n \leq 5$,the graph K_n is difference cordial only when $n \leq 4$ while the bipartite graph $K_{m,n}$ is not difference cordial if $m \geq 4$ and $n \geq 4$,the bistar $B_{m,n}$ is not difference cordial when $m+n \geq 9$ but the wheel W_n,the fan F_n,the gear G_n,the helm H_n and all webs are difference cordial graphs for all n [4]. In [5] the authors investigated the difference cordial labeling behavior of $G \odot P_n, G \odot mK_1$ $(m = 1, 2, 3)$ where G is either unicyclic or a tree and $G_1 \odot G_2$ are some more standard graphs. Some graphs obtained from triangular snake and quadrilateral snake were investigated with respect to the difference cordial labeling behavior. Also the behavior of subdivision of some snake graphs is investigated in [5].

Proposition 1.2 *If G is a (p, q) difference cordial graph,then $q \leq 2p - 1$ [4].*

Definition 1.3 *The number $\delta(G) = \min\{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max\{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, the number $d(G) = \frac{1}{|V|}\sum\limits_{v \in V} d(v)$ is the average degree of the vertices in the graph G [7]*

Definition 1.4 *A fan graph is obtained by joining all vertices of a path P_n to a further vertex,called the center. Thus F_n contains $n+1$ vertices say $c, v_1, v_2, v_3, ..., v_n$ and $2n-1$ edges,say $cv_i, 1 \leq i \leq n$, and,$v_iv_{i+1}, 1 \leq i \leq n-1$.*

Notation 1.5 *The maximum number of edges labelled 1 that is related with a specific vertex,equals 2.*

2 Main Results

Proposition 2.1 *The graph $G(p, q)$ is not difference cordial graph if $\delta(G) \geq 4$.*

Proof. Let $G(p, q)$ be any graph with $\delta(G) \geq 4$; then the minimum value of q is $2p$; but $2p \nleq 2p - 1$, this contradicts Proposition 1.2. ■

Proposition 2.2 *The graph $G(p,q)$ is not difference cordial if $d(G) \geq 4$.*

Proof. Let $G(p,q)$ be any graph with $d(G) \geq 4$;then the value of q is more than or equal to $2p$, but $2p \nleq 2p-1$, which is contradicts Proposition 1.2. ■

Remark 1 *The value of $e_f(0)$ is not exceeding p in any difference cordial graph $G(p,q)$.*

Proof. Direct consequence of Proposition 1.2. ■

Proposition 2.3 *Let $G(p,q)$ be a graph with two vertices of degree $(p-1)$ then G is not a difference cordial graph for all $p \geq 8$.*

Proof. Let $G(p,q)$ be a graph with p vertices, $p \geq 8$ and has two vertices v_i, v_j of degree $(p-1)$ then there are $2p-3$ different edges incident with them, If there are more than two additional edges then G is not difference cordial since $q \nleq 2p-1$. If there are only two additional edges then $q = 2p-1$, then we have two cases:

Case 1: the edge connecting v_i and v_j is labelled 0, then there are at most 6 edges labelled 1: two passing through v_i, two are passing through v_j and the two additional edges. In this case

$$|2p-7-6| = |2p-13| \geq 2 \textit{ where } p \geq 8$$

i.e., G is not difference cordial .

Case 2: the edge connecting v_i and v_j is labelled 1, then there are at most 5 edges labelled 1: one passing through v_i and v_j, two edges are: one is incident with v_i and other is incident with v_j and the two additional edges. In this case

$$|2p-6-5| = |2p-11| \geq 2 \textit{ where } p \geq 7$$

i.e., G is not difference cordial.

In case there is one additional edge, other than those incident with v_i, v_j, similar argument is used . ■

Example 2.4

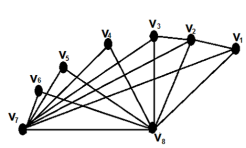

Figure 1: The graph $G = (8,15)$

$\deg(V_8) = 7, \deg(V_7) = 7$ *and G cannot be a difference cordial graph.*

Proposition 2.5 *Let $G(p, q)$ be any graph with two vertices of degrees $(p-1)$ and $(p-2)$; then G is not a difference cordial graph for all $p \geq 9$.*

Proof. Similar to the proof of Proposition 2.3. ■

Example 2.6

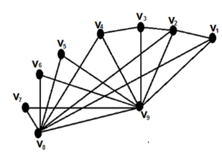

Figure 2: The graph $G = (9, 17)$

$\deg(V_8) = 7, \deg(V_9) = 8$ *and G cannot be a difference cordial graph.*

In [6] theorem 2.14 ,R. Ponraj,S. Sathish Narayanan and R. Kala state that "Let G be a (p, q) difference cordial graph with $k(k > 1)$ vertices of degree $p-1$. Then $p \leq 7$ ". However :

Corollary 2.7 *The graph $G(p, q)$ is not a difference cordial graph if there exist three vertices of degree $(p-1)$ for all $p \geq 6$.*

Proof. Let $G(p, q)$ be a graph with three of its vertices of degree $p-1$ then there exist at least $3p-6$ edges in the graph, by Proposition 2 if the graph is a difference cordial graph then

$$3p - 6 \leq 2p - 1$$

A contradiction when $p \geq 6$. ■

Example 2.8

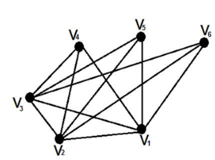

Figure 3: The graph $G = (6, 12)$

$12 \nleq 2 * 6 - 1$ *and G cannot be a difference cordial graph.*

Proposition 2.9 *Let G be a (p, q) graph with one vertex of degree $(p-1)$ then G is not a difference cordial if there exists a set of non adjacent vertices S with* $\sum_{v_i \in S}(\deg(v_i) - 3) \geq 4$.

Proof. Let G be a (p, q) graph with p vertices and have a vertex v_k of degree $p-1$ and there exists a set of non adjacent vertices S with$\sum_{v_i \in S}(\deg(v_i)-3) \geq 4$. Then there are at least $p-3$ edges passing through v_k labelled 0, hence $e_f(0) \geq p-3+4 = p+1$, i.e.,G is not a difference cordial graph. ■

Example 2.10

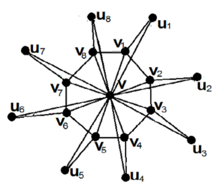

Figure 4: The flower graph Fl_8

$p = 17, q = 32$
$deg(v) = 16$
$S = \{v_1, v_3, v_5, v_7\}$ *then* $\sum_{v_i \in S}(\deg(v_i) - 3) = 1+1+1+1 = 4$ *there are at least* $4 + 14 = 18$ *edges labelled* 0 *then the graph is not a difference cordial.*

Proposition 2.11 *Let G be a (p, q) graph then G is not difference cordial graph if there exists a set of non adjacent vertices S with* $\sum_{v \in S}(\deg(v) - 2) = p + 1$.

Proof. Let S be a set of non adjacent vertices with $\sum_{v_i \in S}(\deg(v_i)-2) = p+1$ Since the maximum number of edges labelled 1 that are incident with a specific vertex equals 2,the number of edges labelled 0 that are incident with vertices of S are at least $\sum_{v_i \in S}(\deg(v_i) - 2)$ this means the minimum value for $e_f(0)$ in the graph G is $p+1$,therefor the graph cannot be a difference cordial graph. ■

Proposition 2.12 *The complement graph of a difference cordial graph is not difference cordial when the number of its vertices is more than eight.*

Proof. Let G be a (p, q) difference cordial graph with $p \geq 9$ then by proposition 1.2.

$$q \leq 2p - 1 \tag{1}$$

G^c, the complement of graph G contains $\frac{1}{2}p(p-1)-q$ edges and p vertices, let G^c be difference cordial then

$$\frac{1}{2}p(p-1)-q \leq 2p-1 \tag{2}$$

by adding (1) and (2) we get

$$\frac{1}{2}p(p-1) \leq 4p-2$$

$$p^2-9p \leq -4$$

A contradiction for all $p \geq 9$ ■

3 Difference cordial labeling for Some graphs:

In This section we will discuss the ability of applying difference cordial labeling for some graphs and the functions which make it difference cordial graphs.

The Proposition 1.2 consider necessary condition for difference cordial labeling but it is not sufficient.

3.1 Ladder graphs L_n

The ladder graph is a planner undirected graph denoted by L_n with $2n$ vertices and $3n-2$ edges [3]. The ladder graph L_n can be expressed as $L_n \cong P_n \times P_2$

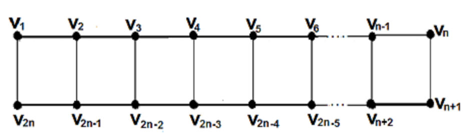

Figure 5: Ladder Graph L_{2n}

Proposition 3.1 *Every ladder graph L_n is difference cordial for all n.*

Proof. Let L_n be a ladder graph, then it has $2n$ vertices and $3n-2$ edges. Let the vertices be $v_1, v_2, ..., v_{2n}$ such that $v_n v_{n+1}$ is an edge in this graph. Define the mapping $f: L_n \longrightarrow \{1, 2, ..., 2n\}$ by :

$$f(v_i) = \left\{ \begin{array}{lll} i & if & 1 \le i \le \lceil \frac{1}{2} |E| \rceil \\ 3 \lceil \frac{1}{2} |E| \rceil + \lceil \frac{1}{2} n \rceil + 2 - 2i & if & \lceil \frac{1}{2} |E| \rceil < i \le \lceil \frac{1}{2} |E| \rceil + \lceil \frac{1}{4} n \rceil \\ 2(i-n) - 1 & if & \lceil \frac{1}{2} |E| \rceil + \lceil \frac{1}{4} n \rceil < i \le 2n \;\; and\ n\ is\ odd \\ 2(i-n) & if & \lceil \frac{1}{2} |E| \rceil + \lceil \frac{1}{4} n \rceil < i \le 2n \;\; and\ n\ is\ even \end{array} \right\}$$

From the first part of definition notice that there are $\lceil \frac{1}{2} |E| \rceil - 1$ of edges labelled 1, in the second part we notice that

$$|f(v_{i+1}) - f(v_i)| = \left| 3 \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2} n \right\rceil + 2 - 2(i+1) - 3 \left\lceil \frac{1}{2} |E| \right\rceil - \left\lceil \frac{1}{2} n \right\rceil - 2 + 2i \right| = 2$$

, so

$$\begin{aligned} \left| f(v_i) - f(v_{2n-(i+1)}) \right| &= \left| 3 \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2} n \right\rceil + 2 - 2i - 2n + (i+1) \right| \\ &= \left| 3 \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2} n \right\rceil + 3 - i - 2n \right| \\ &= \left| 3 \left\lceil \frac{1}{2} (3n-2) \right\rceil + \left\lceil \frac{1}{2} n \right\rceil + 3 - i - 2n \right| \\ &= |3n - i\ | > 1 \end{aligned}$$

which means all these edges are labelled 0. In the third part of definition we notice when n is even :

$$|f(v_{i+1}) - f(v_i)| = |2(i+1-n) - 2(i-n)| = 2$$

and

$$\begin{aligned} \left| f(v_i) - f(v_{2n-(i+1)}) \right| &= |2(i-n) - 2n + (i+1)| = |3i - 4n| \\ &> \left| 3(\left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{4} n \right\rceil) - 4i \right| \\ &> \left| 3(\left\lceil \frac{1}{2} (3n-2 \right\rceil + \left\lceil \frac{1}{4} n \right\rceil) - 4n \right| \\ &> \left| \frac{1}{2} n + 3 \left\lceil \frac{1}{4} n \right\rceil - 3 \right| \\ &> \left\{ \begin{array}{lll} |5m-3| & if & n = 4m \\ & & \\ |5m+1| & if & n = 4m \end{array} \right\} \\ &> 2 \end{aligned}$$

this means all the edges $v_i v_{2n-(i+1)}$ in this third part are labelled 0. But if n is an even number then the number of the total edges of the ladder L_n is even and thus there must exist additional edge labelled 1, which we may get it from the label of the last vertex in part two and the first label in part three.

Notice that if $i = \lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil$ then

$$f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil}) = 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) \tag{3}$$

and if $i = \lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1$, then

$$f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}) = 2\left(\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1\right) - n\right) \tag{4}$$

by subtracting (4) from (3) we get

$$\begin{aligned}
&f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil}) - f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}) \\
&= 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) - 2\left(\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1\right) - n\right) \\
&= -\left\lceil \frac{1}{2}|E| \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = -\left\lceil \frac{1}{2}(3n-2) \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil \\
&= \frac{-3}{2}n + 1 + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = n + 1 - 4\left\lceil \frac{1}{4}n \right\rceil = \left\{ \begin{array}{ll} 1 & if\ n = 4m \\ -1 & if\ n = 4m+2 \end{array} \right\}
\end{aligned}$$

thus the edge $v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil} v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}$ is labelled 1, then the graph is difference cordial.

Now if n is an odd number then $|E|$ is an odd number and then from the first part we get $\lfloor \frac{1}{2}|E| \rfloor$ edges labelled 1 and all other edges in the second and third part are labelled 0, similarly when n is even, and

$$\begin{aligned}
&f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil}) - f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}) \\
&= 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) - 2\left(\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1\right) - n\right) + 1 \\
&= -\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil - 4\left\lceil \frac{1}{4}n \right\rceil + 2n - 1 = -\left\lceil \frac{1}{2}(3n-2) \right\rceil + \left\lceil \frac{1}{2}n \right\rceil - 4\left\lceil \frac{1}{4}n \right\rceil + 2n - 1 \\
&= \left\{ \begin{array}{ll} -\lceil \frac{1}{2}(3(4m+1)-2) \rceil + \lceil \frac{1}{2}(4m+1) \rceil - 4\lceil \frac{1}{4}(4m+1) \rceil + 2(4m+1) - 1 & if\ n = 4m+1 \\ -\lceil \frac{1}{2}(3(4m+3)-2) \rceil + \lceil \frac{1}{2}(4m+3) \rceil - 4\lceil \frac{1}{4}(4m+3) \rceil + 2(4m+3) - 1 & if\ n = 4m+3 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} 0 & if\quad n = 4m+1 \\ 1 & if\quad n = 4m+3 \end{array} \right\}
\end{aligned}$$

then

$$\begin{array}{ll} e_f(1) = e_f(0) & if \quad n \ is \quad even \\ e_f(1) = e_f(0) \ -1 & if \quad n \ is \quad odd \ \& \ n = 4m+1 \\ e_f(1) = e_f(0) + 1 & if \quad n \ is \quad odd \ \& \ n = 4m+3 \end{array}$$

Hence G is difference cordial. ■

Example 3.2 *Consider the graph* L_{10}
$n = 10, |E| = 28, \lceil \frac{1}{2}|E| \rceil = 14, \lceil \frac{1}{2}n \rceil = 5, \lceil \frac{1}{4}n \rceil = 3$ *then*

$$f(v_i) = \left\{ \begin{array}{lll} i & if & 1 \le i \le 14 \\ 49 - 2i & if & 14 < i \le 17 \\ 2(i-9) & if & 17 < i \le 20 \end{array} \right\}$$

$f(v_1) = 1, f(v_2) = 2, ..., f(v_{14}) = 14,$
$f(v_{15}) = 19, f(v_{16}) = 17, f(v_{17}) = 15,$
$f(v_{18}) = 16, f(v_{19}) = 18, f(v_{20}) = 20.$ $e_f(0) = 14, e_f(1) = 14$

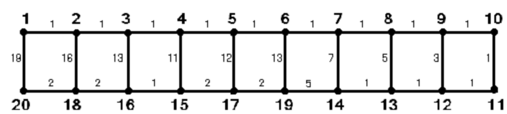

Figure 6: A difference cordial labeling for L_{10}

Example 3.3 *Consider the graph* L_{11} $n = 11, |E| = 31, \lceil \frac{1}{2}|E| \rceil = 16, \lceil \frac{1}{2}n \rceil = 6, \lceil \frac{1}{4}n \rceil = 3$ *then*

$$f(v_i) = \left\{ \begin{array}{lll} i & if & 1 \le i \le 16 \\ 56 - 2i & if & 16 < i \le 19 \\ 2(i-n) - 1 & if & 19 < i \le 22 \end{array} \right\}$$

$f(v_1) = 1, f(v_2) = 2, ..., f(v_{16}) = 16,$
$f(v_{17}) = 22, f(v_{18}) = 20, f(v_{19}) = 18,$
$f(v_{20}) = 17, f(v_{21}) = 19, f(v_{22}) = 21.$

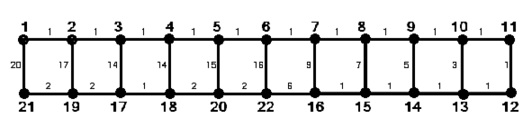

Figure 7: A difference cordial labeling L_{11}

$e_f(0) = 15, \ e_f(1) = 16$

3.2 Triangular ladder graph TL_n

A triangular ladder $TL_n, n \geq 2$, is a graph obtained from the ladder $L_n = P_n \times P_2$ by adding the edges $u_i v_{i+1} for 1 \leq i \leq n-1$. Such graph has $2n$ vertices with $4n-3$ edges

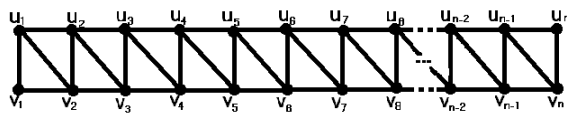

Figure 8: Triangle ladder graph TL_n

Proposition 3.4 *The triangular ladder graph $TL_n, n \geq 2$ is a difference cordial graph for all n.*

Proof. Let $G = TL_n, n \geq 2$ be a triangular ladder graph ,then $G = (2n, 4n-3)$.

Define the function

$$f(v_i) = 2i - 1 \;\; and \;\; f(u_i) = 2i \; ; \; 1 \leq i \leq n \tag{5}$$

It is clear that $e_f(1) = 2n-1$ hence $e_f(0) = (4n-3) - (2n-1) = 2n-2$, then $|e_f(0) - e_f(0)| = 1$,

thus $G = TL_n, n \geq 2$ is a difference cordial graph. ■

Example 3.5 *Consider the graphs TL_6 and TL_7*

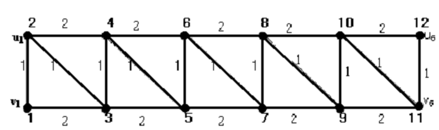

Figure 9: A difference cordial labeling for TL_6

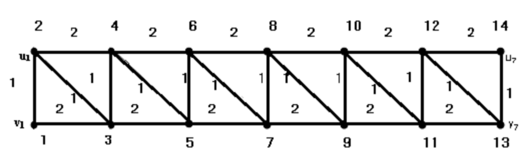

Figure 10: A difference cordial labeling for TL_7

3.3 The Grid graph $P_m \times P_n$

In this subsection we will investigate the difference cordial labeling for every grid graph of the form $P_m \times P_n$ for all m, n . Let the vertices of the grid graph be arranged as a sequence in certain order as in the figure 11

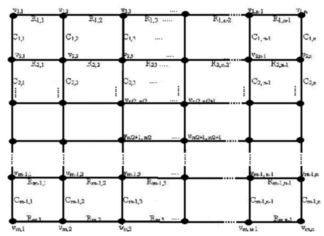

Figure 11: The grid graph $P_m \times P_n$

This kind of graphs contains mn vertices and $2mn - (m + n)$ edges.

Proposition 3.6 *Every grid graph is $P_m \times P_n$ is a difference cordial graph for all integers m, n greater than* 1.

Proof. Let G be a graph $P_m \times P_n$ then $G = (mn, 2mn - (m + n))$ **Case 1:** If $m = n$ then $|V| = n^2$ and $|E| = 2(n^2 - n)$, define the function f for labeling vertices of G by :

$$f(v_{ij}) = (i - 1)\, n + j$$

in each row of the grid graph there exist $n - 1$ edges labelled 1 this leads to $e_f\,(1) = n(n - 1)$ and the number of edges labelled 0 is equal to :

$$2n(n - 1) - n(n - 1) = n(n - 1),$$

thus G is a difference cordial graph.

Case 2 : If $|m - n| = 1$, then $|V| = mn$ and $|E| = 2mn - (m + n)$. Let $n = m + 1$ then $|E| = 2m^2 - 1$. Now using the same functions in **Case 1** we will get

$$e_f(1) = m(n - 1) = m(m + 1 - 1) = m^2$$

and

$$e_f(0) = (m - 1)(m + 1) = m^2 - 1$$

which means the graph is a difference cordial graph. Similarly if $m = n + 1$ **Case 3 :**If $|m - n| \geq 2$. Let $n > m$ and let $k = \left\lceil \frac{1}{2}(n - m) \right\rceil$ we define the mapping :

$$f(v_{ij}) = \left\{ \begin{array}{lll} (j-1)\, m + i & if & 1 \leq j \leq k \\ k\,(m-1) + n\,(i-1) + j & if & j = k+1, ..., n \end{array} \right\}$$

It follows that :

$$\begin{aligned} e_f(1) &= k(m-1) + m(n-k-1) \\ &= mn - (m+k) \end{aligned}$$

and

$$\begin{aligned} e_f(0) &= 2mn - (m+n) - mn + (m+k) \\ &= mn - (n-k) \end{aligned}$$

so

$$\begin{aligned} |e_f(0) - e_f(1)| &= |mn - n + k - mn + m + k| \\ &= |-n + 2k + m| \\ &= \left\{ \begin{array}{lll} 0 & if\ \ n-m\ \ is\ \ even \\ 1 & if\ \ n-m\ \ is\ \ odd \end{array} \right\} \end{aligned}$$

Similarly if $m > n$ we apply the same mapping but replacing i by j and m by n, i.e.:
$k = \left\lceil \frac{1}{2}(m-n) \right\rceil$ and :

$$f(v_{ij}) = \left\{ \begin{array}{ll} (i-1)\,n + j & if\quad 1 \le i \le k \\ k\,(n-1) + m\,(j-1) + i & if\quad i = k+1\ ,\ ...\ , m \end{array} \right\}$$

Hence the grid graph $P_m \times P_n$ is a difference cordial graph for all m, n. ■

Example 3.7 *Let* $P_m \times P_n = P_4 \times P_3$
$n = 3, m = 4, |V| = 12, |E| = 17$
$f(v_{ij}) = 3(i-1) + j$

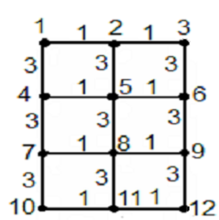

Figure 12: A difference cordial labeling for grid graph $P_4 \times P_3$

$e_f(1) = 8, e_f(0) = 9$

Example 3.8 *Let* $P_m \times P_n = P_5 \times P_8$
$n = 8, m = 5, |V| = 40, |E| = 67, k = 2$

$$f(v_{ij}) = \left\{ \begin{array}{ll} 5(j-1) + j & 1 \le j \le 2 \\ 2(5-i) + 8(i-1) + j & j > 2 \end{array} \right\}$$

$e_f(0) = 34\ \ , e_f(1) = 33$

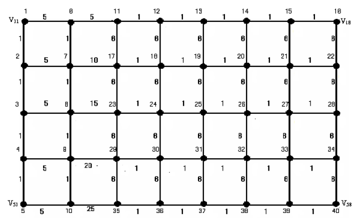

Figure 13: A difference cordial labeling for grid graph $P_5 \times P_8$

3.4 Step ladder graph $\mathbf{S(T_n)}$:

Definition 3.9 *Let P_n be a path on n vertices denoted by $(1,1),(1,2),...,(1,n)$ and $n-1$ edges denoted by $e_1, e_2, ..., e_{n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i,$i = 1,2,...,n-1$ we erect a ladder with $n-(i-1)$ steps including the edge e_i. The graph obtained is called a step ladder graph and is denoted by $S(T_n)$, where n denotes the number of vertices in the base.*

The following sketch shows the step ladder graph :

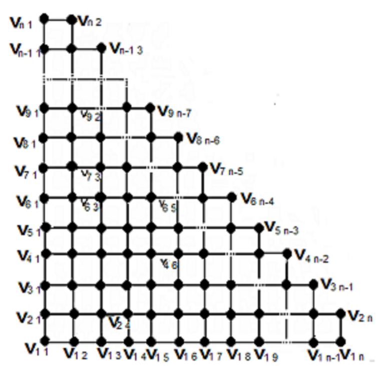

Figure 14: The step ladder graph $S(T_n)$

The number of vertices and edges are :

$$\begin{aligned}|V| &= 2+3+4+...+n+n \\ &= \frac{1}{2}n(n+1)+(n-1) \\ &= \frac{n^2+3n-2}{2}\end{aligned}$$

$$|E| = 2(|V| - n)$$
$$= n(n+1) - 2$$

We notice for all step ladder graphs that $i + j \leq n + 2$

Proposition 3.10 *Every step ladder graph $S(T_n)$ is a difference cordial graph for all n.*

Proof. Let $S(T_n)$ be a step ladder graph then $|E| = n(n+1) - 2 = n^2 + n - 2$
Define the function $f : S(T_n) \longrightarrow \{1, 2, ..., \frac{1}{2}n(n+1) + (n-1)\}$ by :

$$f(v_{ij}) = \begin{cases} j + (i-1)n & 1 \leq i \leq 3 \\ j + (i-1)n - \frac{1}{2}(i-3)(i-2) & i \geq 4 \end{cases}$$

$$\begin{aligned} e_f(1) &= (3n-4) + (n-3) + (n-4) + (n-5) + ... + 3 + 2 + 1 \\ &= (n-1) + (n-1) + (n-2) + (n-3) + \ ... \ + 2 + 1 \\ &= (n-1) + \frac{1}{2}n(n-1) \\ &= \frac{1}{2}(n^2 + n - 2) \ , \end{aligned}$$

then $e_f(1) = \frac{1}{2}|E|$ which means $|e_f(1) - e_f(0)| = 0$.
Therefor $S(T_n)$ is a difference cordial graph for all n ■

3.5 Double Sided Step Ladder Graph $2S(T_{2n})$:

Definition 3.11 *Let P_{2n} be a path of length $2n-1$ with $2n$ vertices $(1,1), (1,2), ..., (1,2n)$ with $2n-1$ edges,$e_1, e_2, ..., e_{2n-1}$, where e_i is the edge joining the vertices $(1,i)$and$(1,i+1)$. On each edge e_i, for $i = 1, 2, ..., n$, we erect a ladder with $i+1$ steps including the edge e_i and on each edge e_i, for $i = n+1, n+2, ..., 2n-1$,we erect a ladder with $2n+1-i$ steps including the edge e_i.The double sided step ladder graph $2S(T_{2n})$ has vertices denoted by $(1,1), (1,2), ..., (1,2n), (2,1), (2,2), ..., (2,2n), (3,2), (3,3), ..., (3,2n-1), (4,3), (4,4), , (4,2n-2), ..., (n+1,n), (n+1,n+1)$. In the ordered pair (i,j), i denotes the row number (counted from bottom to top) and j denotes the column number (from left to right) in which the vertex occurs.*

Example 3.12 *The figure 15 is the $2S(T_{10})$*

Proposition 3.13 *The double sided step ladder graph $2S(T_m)$ is a difference cordial graph, where $m = 2n$ denotes the number of vertices in the base.*

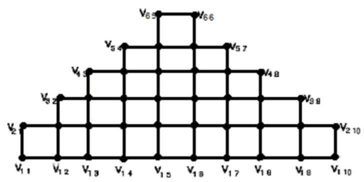

Figure 15: Double sided step ladder graph $2S(T_{2n})$

Proof. Let $G=(V,E)$ be the double sided step ladder graph $2S(T_m)$ where $m=2n$ then

$$|V| = n^2+3n \qquad and \qquad |E| = 2n^2+3n-1$$

Define $f: V \longrightarrow \{1,2,...,n^2+3n\}$ by :

$$f(v_{i,j}) = \left\{ \begin{array}{cc} u & if \quad i=1 \ \ and \ \ j \leq \left\lceil \frac{1}{2}n \right\rceil \\ j+2n\ (i-1) & if \ \ i=1 \ and \ \ j \geq \left\lceil \frac{1}{2}n \right\rceil + 1 \\ j+2n\ (i-1) & if \ \ i=2 \\ j+2n\ (i-1)-(i-1)^2 & if \ \ i=3,4,...,n+1 \end{array} \right\}$$

where

$$u = \left\{ \begin{array}{cc} 2j \ (\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) & if \ \ n=3 \ or \ n \equiv 0 \pmod 4 \\ 2j \ (\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) \ + \left\lfloor \frac{2j}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor & if \quad n \equiv 1,2 \pmod 4 \\ (2j-1) \ (\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) \ + 2 \left\lfloor \frac{2j}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor & if \quad n \equiv 3 \pmod 4 \end{array} \right\}$$

from the last three parts of the definition of f we will get $n^2+2n-\left\lceil \frac{1}{2}n \right\rceil - 1$ edges give $e_f(1)$, while in the first part all edges give $e_f(0)$ except when $\left\lceil \frac{1}{2}n \right\rceil \leq 4$ we will get an edge in $e_f(1)$ since $1 \leq u \leq 3$.

Case 1 : If $n=2$ then $e_f(1) = n^2+2n-\left\lceil \frac{1}{2}n \right\rceil -1+1 = 7$ and $e_f(0)=6$,
if $n=3$ then $e_f(1) = n^2+2n-\left\lceil \frac{1}{2}n \right\rceil -1+1 = 13$ and $e_f(0)=13$,
if $n=4$ then $e_f(1) = n^2+2n-\left\lceil \frac{1}{2}n \right\rceil -1+1 = 22$ and $e_f(0)=21$ and
if $n=5$ then $e_f(1) = n^2+2n-\left\lceil \frac{1}{2}n \right\rceil -1+1 = 32$ and $e_f(0)=32$.

Case 2 : If $n \equiv 0 (\bmod 4)$ then $n = 4k$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k$, then $|E| = 2(4k)^2 + 3(4k) - 1 = 32k^2 + 12k - 1$ and

$$\begin{aligned} f(v_{1\lceil \frac{1}{2}n \rceil}) &= 2(\left\lceil \frac{1}{2}n \right\rceil) \ (\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) \\ &= (2 * 2k)(\bmod 2k+1) = 4k(\bmod 2k+1) \\ &= 2k - 1 \end{aligned}$$

while $f(v_{1\ \lceil \frac{1}{2}n \rceil + 1}) = \lceil \frac{1}{2}n \rceil + 1 = 2k + 1$,

thus the label of the edge $v_{1\lceil \frac{1}{2}n \rceil} v_{1\ \lceil \frac{1}{2}n \rceil + 1}$ will be included in $e_f(0)$,

therefor $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 = 16k^2 + 8k - 2k - 1 = 16k^2 + 6k - 1$

and $e_f(0) = |E\ | - e_f(1) = 32k^2 + 12k - 1 - 16k^2 - 6k + 1 = 16k^2 + 6k$ we get $|e_f(0) - e_f(1)| = 1$.

Case 3 : If $n \equiv 1 (\bmod 4)$ then $n = 4k+1$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k + 1$, then $|E| = 2(4k+1)^2 + 3(4k+1) - 1 = 32k^2 + 28k + 4$ and

$$\begin{aligned} f(v_{1\lceil \frac{1}{2}n \rceil}) &= 2(\ \left\lceil \frac{1}{2}n \right\rceil) \ (\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) + \left\lfloor \frac{2 \lceil \frac{1}{2}n \rceil}{\lceil \frac{1}{2}n \rceil + 1} \right\rfloor \\ &= 2(2k+1) \ \ (\bmod \ \ 2k+2) + \left\lfloor \frac{2 \lceil 2k+1 \rceil}{\lceil 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+2) \ \ (\bmod \ \ 2k+2) + 1 \\ &= 2k + 1 \end{aligned}$$

while $f(v_{1\ \lceil \frac{1}{2}n \rceil + 1}) = \lceil \frac{1}{2}n \rceil + 1 = 2k + 2$,

thus the label of the edge $v_{1\ \lceil \frac{1}{2}n \rceil} v_{1\lceil \frac{1}{2}n \rceil + 1}$ will be included in $e_f(1)$,

therefor $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 + 1 = (4k+1)^2 + 2(4k+1) - 2k - 1 - 1 + 1 = 16k^2 + 14k + 2$

and $e_f(0) = |E| - e_f(1) = 32k^2 + 28k + 4 - 16k^2 - 14k - 2 = 16k^2 + 14k + 2$ we get $|e_f(0) - e_f(1)| = 0$

Case 4 : If $n \equiv 2 \ (\bmod 4)$ then $n = 4k+2$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k + 1$, then$|E| = 2(4k+2)^2 + 3(4k+2) - 1 = 32k^2 + 44k + 13$ and

$$\begin{aligned} f(v_{1\ \lceil \frac{1}{2}n \rceil}) &= (\ 2 \left\lceil \frac{1}{2}n \right\rceil)(\bmod \left\lceil \frac{1}{2}n \right\rceil + 1) + \left\lfloor \frac{2 \lceil \frac{1}{2}n \rceil}{\lceil \frac{1}{2}n \rceil + 1} \right\rfloor \\ &= 2(2k+1) \ (\bmod \ \ 2k+2) + \left\lfloor \frac{2 \lceil 2k+1 \rceil}{\lceil 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+2) \ \ (\bmod \ \ 2k+2) + 1 \\ &= 2k + 1 \end{aligned}$$

while $f(v_{1\ \lceil\frac{1}{2}n\rceil+1}) = \lceil\frac{1}{2}n\rceil+1 = 2k+2$, thus the label of the edge $v_{1\lceil\frac{1}{2}n\rceil}v_{1\lceil\frac{1}{2}n\rceil+1}$ will included in $e_f(1)$,
therefore $e_f(1) = n^2+2n-\lceil\frac{1}{2}n\rceil-1+1 = (4k+2)^2+2(4k+2)-2k-1-1+1 = 16k^2+22k+7$
and $e_f(0) = |E\ | - e_f(1) = 32k^2+44k+13-16k^2-22k-7 = 16k^2+22k+6$
we get $|e_f(0)-e_f(1)| = 1$
Case 5 : If $n \equiv 3(\bmod 4)$ then $n = 4k+3$ for some positive integer number k and $\lceil\frac{1}{2}n\rceil = 2k+2$, then$|E| = 2(4k+3)^2+3(4k+3)-1 = 32k^2+60k+26$ and

$$\begin{aligned} f(v_{1\ \lceil\frac{1}{2}n\rceil}) &= (\ 2\left\lceil\frac{1}{2}n\right\rceil - 1)\ (\bmod \left\lceil\frac{1}{2}n\right\rceil + 1) + 2\left\lfloor\frac{2\lceil\frac{1}{2}n\rceil-1}{\lceil\frac{1}{2}n\rceil+1}\right\rfloor \\ &= (2(2k+2)-1)(\bmod 2k+3) + 2\left\lfloor\frac{2\lceil 2k+1\rceil-1}{\lceil 2k+1\rceil+1}\right\rfloor \\ &= (4k+3)(\bmod 2k+3)+2 \\ &= 2k+2, \end{aligned}$$

while $f(v_{1\lceil\frac{1}{2}n\rceil+1}) = \lceil\frac{1}{2}n\rceil+1 = 2k+3$, thus the label of the edge $v_{1\ \lceil\frac{1}{2}n\rceil}v_{1\lceil\frac{1}{2}n\rceil+1}$ will be included in $e_f(1)$,
therefor $e_f(1) = n^2+2n-\lceil\frac{1}{2}n\rceil-1+1 = (4k+3)^2+2(4k+3)-2k-2-1+1 = 16k^2+30k+13$
and $e_f(0) = |E| - e_f(1) = 32k^2+60k+26-16k^2-30k-13 = 16k^2+30k+13$
we get $|e_f(0)-e_f(1)| = 0$.

From the cases 1,2,3,4and 5 we conclude that the double sided step ladder graph $2S(T_{2n})$ is a difference cordial graph for all integer number n ■

We discuss here some types of graphs not always difference cordial such as diagonal ladder graph ,diagonal grid graph and friendship graph .

Diagonal ladder graph is a ladder with additional edges u_iv_{i+1} and $u_{i+1}v_i$, denoted by DL_n, where n is half its vertices and the number of its edges is $5n-4$.

Corollary 3.14 *The diagonal ladder graphs are difference cordial graphs if* $n \leq 3$.

Proof. Let the graph G be the diagonal ladder graph DL_n with $2n$ vertices that means there are $5n-4$ edges in G, G is a difference cordial graph. Then we get by Proposition 2

$$\begin{aligned} 5n-4 &\leq 2(2n)-1 \\ n &\leq 3 \end{aligned}$$

then the diagonal ladder graph is difference cordial when $n = 2$ or$n = 3$ ■

The following example shows that DL_2 and DL_3 are difference cordial

Example 3.15 *The following are labeling for the diagonal ladder graphs* DL_2, DL_3

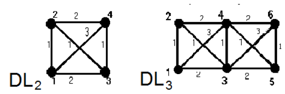

Figure 16: The difference cordial labelings for the diagonal ladder graphs DL_2 & DL_3

The graph $P_m \times P_n$ with diagonal edges is called diagonal grid graph and denoted by $D(\ P_m \times P_n)$. It has mn vertices and $2(2mn+1)-3(m+n)$ edges.

Remark 2 *Diagonal grid graph* $P_m \times P_n$ *are not difference cordial graphs for both* $m, n \geq 3$.

Proof. Let $G = D\,(P_m \times P_n)$, then from Proposition 2 if G is a difference cordial graph then $q \leq 2p-1$. Let $m = n = 3$, then

$$\begin{aligned} q &= 2(2mn+1) - 3(m+n) \\ &= 2(2 \cdot 3 \cdot 3 + 1) - 3(3+3) \\ &= 20 \nleq 17 \end{aligned}$$

then $D\,(P_m \times P_n)$ cannot be a difference cordial graph for both$m, n \geq 3$ ■

This is consistent with corollary 3.5 since diagonal ladder graphs are diagonal grid graphs.

Another type of graphs will be discussed here named one-point union fan graph, where a graph G in which a vertex distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph,the Graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called a one-point union of the n copies of G.

Proposition 3.16 *The fan graph* F_n *is difference cordial for all* n. *citepon1*

Proposition 3.17 *The one-point union* $F_n^{(m)}$ *of* m *copies of a fan* F_n *is difference cordial for all* n *and for* $m \leq 5$.

Proof. Let $G = F_n^{(m)}$, then $|V(G)| = mn+1$ and $|E(G)| = m(2n-1)$. These vertices are : v_{00} is the central vertex and the other vertices are denoted by $v_{ij}, 1 \leq i \leq nand1 \leq j \leq m$

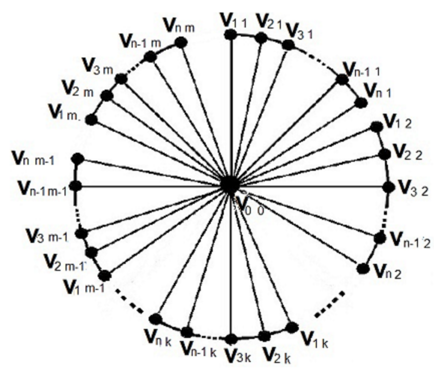

Figure 17: The graph $F_n^{(m)}$

For each copy of a fan F_n there are $n-1$ edges labelled 1, therefore there are $m(n-1)+2$ edges labelled 1 in $F_n^{(m)}$, where the central vertex is labelled $1(\operatorname{mod} n)$ but is neither 1 nor $mn+1$ then

$$e_f(0) = m(2n-1) - m(n-1) - 2 = mn - 2$$

Now

$$\begin{aligned} &|e_f(0) - e_f(1)| \\ &= |mn - 2 - m(n-1) - 2| \\ &= |m - 4| \end{aligned}$$

then $|e_f(0) - e_f(1)| \geq 2$ for all$m \geq 6$. We define the mapping f for $m \leq 5$ and $n \in \mathbb{N}$ by

$$f(v_{0\ 0}) \equiv 1(\operatorname{mod} n) and\ f(v_{0\ 0}) \neq 1, mn+1$$

and

$$f(v_{ij}) = \left\{ \begin{array}{lll} (j-1)n+i & if & (j-1)n+i < f(v_{0\ 0}) \\ \\ (j-1)n+i+1 & if & (j-1)n+i > f(v_{0\ 0}) \end{array} \right\}$$

for all $i, j, 1 \leq i \leq n, 1 \leq j \leq m$. ■

As a special case,the friendship graph denotes by $F_2^{(m)}$ consists of one vertex union with m copies of paths P_2 consisting of $2m+1$ vertices and $3m$ edges as shown in Figure 18

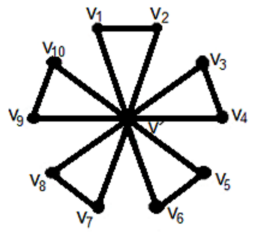

Figure 18: The friendship graph F_5

Therefore the friendship graph $F_2^{(m)}$is difference cordial if and only if $m \leq 5$.

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