Ain Shams University

On Difference Cordial Graphs and Other Graphs

Thesis by

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Submitted to

Department of mathematics - Faculty of Science

Ain Shams University - Cairo - Egypt

for the Degree of Doctor of Philosophy

in Pure Mathematics

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In the name of Allah, the most merciful, the most compassionate all praise be to Allah, the Lord of the worlds; and prayer and peace be upon his servant and messenger Mohamed alayhi wa-alehe wa-sallam.
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Shakir Al-Azzawy

Abstract

Graph labeling is one of the important branches of Graph Theory and became a principal tool in many applications on different sciences and technologies. All that leads to appearance of more than one type of labeling and multiple techniques to meet the required purposes.

In this thesis we study the two main types of graph labeling and introduce the labelings for interested families of graphs and a tractive results for graphs of these types. We state some basic definitions and theorems in graph theory which we need. We divide the other work into four chapters:

In chapter two we introduce some results in difference cordial graphs and difference cordial labelings for some families of graphs such as: ladder, triangular ladder, grid, step ladder and two sided step ladder graph. Also we discussed some families of graphs which may be difference cordial or not, such as diagonal ladder and some types of one-point union of graphs.

In chapter three we introduce some results on difference cordial graphs, where we present results concerning the relation between difference cordiality and the lengths of paths on graphs and study the Semi-Hamiltonian graph, biconnected outerplanar graphs and the line graph of

a graph. Also, we describe the difference cordial labeling for some families of graphs such as: the graph obtained by duplication a vertex by an edge, bow graphs, butterfly graphs, shell-flower graphs and one-point union of complete graphs.

In chapter four we introduce some results on divisor cordial graphs and describe the divisor cordial labeling for the families of graphs: the jelly fish graph, the shell, the bow graph, butterfly graphs and the friendship graphs. In the last chapter we introduce results in divisor cordial labeling for regular graphs, divisor labelings for all graphs with number of vertices less than eight, and divisor cordial labelings for some types of trees such as: olive trees, spider trees, $m-$star trees, $k-$distant trees, caterpillar trees and banana trees.

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Summary

This thesis sheds light on the two concepts of types of graph labeling and describe the labeling for many families of graphs.

Graph labeling is one of the famous problems in Graph Theory. Recently graph labeling became more important because the growth of its applications in many of sciences and technology on a different area such as: computer programming, coding theory, neural network, biotechnology, in the study of X-Ray crystallography, radar, communication network, circuit layouts. In this work by a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Harary [14] and for graph labeling, Gallian [12] is referred to.

In general Graph labeling is a strong communication between Number theory and structure of graphs. Nowadays nearly 200 graph labelings techniques have been studied.

Throughout this work we present new results in two types of graph labelings, and discuss the labeling of many kinds of graphs in chapters $2, 3, 4$ and 5.

In **chapter two**: the basic definitions and theorems of graph theory are introduced which are useful in current work, and the outline of the thesis.

In year 2013, Ponraj, Shathish Naraynan and Kala introduce the notions of difference cordial labeling for finite undirected and simple graph. In chapter two, we present some new results on difference cordial graphs under a title "On Difference Cordial Graphs" which are published in the mathematical Bulgarian journal "Mathematica AEterna" journal. This chapter comprises four sections and present new interesting results and facts in difference cordial graphs:

- Seven results concerning the degree of vertices and difference cordiality.
- One result concerning the graph and its complements.

In addition, we describe the function of labeling for different families of graphs such as: ladder, step ladder, two sided step ladder, diagonal ladder, triangular ladder, grid graph and some types of one-point union of graphs. All that appears in:

Mathematica Aeterna, Vol. 5, 2015, no. 1, $105 - 124$. [37]

In **chapter three**: Some new results and examples on difference cordial graphs, and interested results about:

- five results about relation between the lengths of disjoint paths in graph and difference cordiality of a graph, in addition, Petersen, semi-Hamiltonian graph and outerplanar graph.

- Two results about line graph.
- A result for union of graphs.

Also we describe difference cordial labelings for the families of graphs: bow, buttery, Shell-Flower and One-Point Union of Complete graphs. These results are published in the academic journal "TURKISH JOURNAL OF MATHEMATICS" in Turkey.

Turk J Math, 40, (2016), 417-427 [38]

By combining the divisibility concept in Number theory and Cordial labeling concept in Graph labeling, Varatharajan, Navanaeethakrishnan Nagarajan in 2011, introduced a new concept called divisor cordial labeling.

In **chapter four**: some new results on divisor Cordial graph labeling are introduced:

new general results in divisor cordial labeling, four results in maximal number of edges are labeled one in any graph and in the regular graph. We introduce and discusses mappings of labelings for some families of graphs such as: jelly fish, shell, bow, butterfly and friendship graph, these results are submitted for publication in the Indian academic journal: Journal of Graph Labeling.

In **chapter five**: new results in divisor cordial labeling for the regular graphs, and divisor cordial labelings for all graphs with number of vertices less than eight except the graph K_4 and proof it not divisor cordial graph. As well the divisor cordial labeling for the trees: olive trees, spider trees, $m-$star trees, $k-$distant trees, caterpillar trees and banana trees.

Chapter 1

Introduction

1.1 Brief Introduction to Labeling

Graph labeling is a strong communication between Number Theory and structure of graphs. It is an assignment of integers to the vertices, edges, or both, subject to certain conditions. Most graph labeling methods extract their origin from a paper introduced firstly by Rosa in 1964 [30]. Diverse types are the subject of much study, where during the last 50 years nearly 200 graph labelings techniques have been studied in over 2000 papers [12].

Graph Labeling is a powerful tool that makes things ease in various fields of computer science, public key cryptography, Networks representation, database management [28].

Most of the graph labeling problems have three ingredients: A set of numbers S from which the labels are chosen; rule that assigns a value to each vertex or edge such that some conditions must be satisfied [12].

The problems related to labeling of graphs challenge our mind for their eventual solutions. Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes, convolution codes with optimal autocorrelation properties, X-ray crystallography, communication network, bio-technology and to determine optimal circuit layouts. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb [5].

A graceful labeling is an assignment of the integers $\{1, 2, ..., n\}$ to vertices of a graph such that once each edge is labeled with difference of its incident vertices, with each integer in $\{1, 2, ..., n-1\}$ is used once and only once. In [30] Rosa has identified essentially three reasons why a graph fails to be graceful:

1. has "too many vertices" and "not enough edges",
2. has too many edges,
3. has the wrong parity.

Rosa [30] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 $(mod\ 4)$ then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Seoud and Abdel-Aal [32] determined all odd-graceful graphs of order at most 6 and proved that if G is odd-graceful then $G \cup K_{m,n}$ is odd-graceful. Seoud and Helmi in [33] proved: if G has an odd-graceful labeling f with bipartition (V_1, V_2) such that:
$max\,\{f(x) : f(x)\ is\ even;\ x \in V_1\}\ <\ min\,\{f(x) : f(x)\ is\ odd;\ x \in V_2\}$,

then G has an $\alpha-$labeling, if G has an $\alpha-$labeling, then $G \odot k_n$ is odd-graceful, and if G_1 has an $\alpha-$labeling and G_2 is odd-graceful, then $G_1 \cup G_2$ is odd-graceful.

They also proved the following graphs have odd-graceful labelings: dragons obtained from an even cycle; graphs obtained from a gear graph by attaching a fixed number of pendent edges to each vertex of degree 2 on rim of the wheel of the graph; $C_{2m} \odot \bar{K_n}$; graphs obtained from an even cycle by attaching a fixed number of pendent edges to every other vertex; graphs obtained by identifying an endpoint of a star $S_n (n \geq 3)$ with a vertex of an even cycle; the graphs consisting of two even cycles of the same order that share a common vertex with any number of pendent edges attached at the common vertex; and the graphs obtained by joining two even cycles of the same order by an edge.

Seoud and Wilson [39] proved that $C_3 \bigcup K_4$, $C_3 \bigcup C_3 \bigcup K_4$ and certain graphs of the form $C_3 \bigcup P_n$ and $C_3 \bigcup C_3 \bigcup P_n$ are not graceful. Seoud and Youssef [40] investigated the gracefulness of specific families of the form $G \bigcup K_{m,n}$. They obtained the following results: $C_3 \bigcup K_{m,n}$ is graceful if and only if $m \geq 2$ and $n \geq 2$; $C_4 \bigcup K_{m,n}$ is graceful if and only if $(m, n) \neq (1, 1)$; $C_7 \bigcup K_{m,n}$ and $C_8 \bigcup K_{m,n}$ are graceful for all m and n; $mK_3 \bigcup nK_{1,r}$ is not graceful for all m, n and r; $K_i \bigcup K_{m,n}$ is graceful for $i \leq 4$ and $m \geq 2$; $n \geq 2$ except for $i = 2$ and $(m, n) = (2, 2)$; $K_5 \bigcup K_{1,n}$ is graceful for all n; $K_6 \bigcup K_{1,n}$ is graceful if and only if n is neither 1 nor 3.

Another best known labeling methods are called harmonious labelings. The harmonious graphs naturally arose in the study by the two researchers Graham and Sloane [13]. They defined a graph G with q

edges to be harmonious if there is an injection f from the vertices of G to the group of integers ($modulo\ q$) such that when each edge xy is assigned the label $f(x) + f(y)(mod\ q)$, the parity condition for harmonious graph the resulting edge labels are distinct.

Seoud and Elsakhawi [36] proved: paths and ladders are arbitrarily graceful; and for $n \geq 3$; K_n is $k-$graceful if and only if $k = 1$ and $n = 3$ or 4.

Seoud and Youssef [41] have shown that the one point union of a triangle and C_n is harmonious if and only if $n \equiv 1\ (mod\ 4)$ and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. They [35] introduced Gracefulness of the union of cycles and paths. Also in [34] Seoud, Abdel Maqsoud, and Sheehan noted that when r or s is even, rC_8 is not harmonious. They proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ are not harmonious when $m+n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has an odd order.

Cordial labeling is a variation of both graceful and harmonious labelings introduced by I. Cahit in 1987 [8].
A-cordial labelings defined as a common generalization of cordial labeling (introduced by Cahit [8]) and harmonious labeling (introduced by Graham and Sloane [13]).

Ponraj, Sathish Narayanan, and Kala [23] introduced the notion of difference cordial labelings.

By combining the divisibility concept in number theory and cordial labeling concept in graph labeling, R. Varatharajan, S. Navanaeethakrishnan, and K. Nagarajan introduce a new concept called divisor cordial labeling [46].

We study and present new results on the last two types of labelings mentioned above. All graphs in this work are simple, finite and undirected.

Graph labelings of diverse types are the subject of much study and the state of the field is described in detail in Gallian dynamic survey [12]. The results obtained so far, while numerous, are mainly piecemeal in nature and lack generality. In an attempt to provide something of a framework for these results, we introduce some of them in the next chapter.

1.2 Some Fundamentals in Graph Theory:

In this section we will describe some of graphs and its properties that we need in our work. Since the language of graph theory is still not standard, all authors have their own terminology.

A graph G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called edges; the use of the word 'family' permits the existence of multiple edges. We call $V(G)$ the vertex set and $E(G)$ the edge family of G. Although we sometimes have to restrict our attention to simple graphs (in any simple graph there is at most one edge joining a given pair of vertices), we shall prove our results for general graphs whenever possible. An edge v, w is said to join the vertices v and w, and is again abbreviated to vw. The number of vertices in G is called the order of G and the number of edges in G is called the size of G. The order and size of G are denoted by p and q respectively, in other word $|V(G)| = p$ and $|E(G)| = q$. A graph is trivial if its vertex set is a singleton.

Let $G = (V, E)$ be a graph. Two vertices v_1 and v_2 are said to be adjacent if there exists an edge $e \in E$,$e = v_1, v_2$; $v \in V$. Two edges e_1 and e_2 are said to be adjacent if there exists a common vertex v on them.

Let $v \in V$, $G = (V, E)$. The neighbors of v are the set of vertices that are adjacent to v. Formally: $N(v) = \{u \in V : e \in E,\ e = u, v\}$.

The degree of a vertex v of a graph G is the number of edges incident to the vertex, with loops counted twice. It is denoted by $deg(v)$, that means $deg(v) = |N(v)|$. The degree sequence of a graph is the sequence formed by arranging the vertex degrees in non - increasing order.

A vertex of degree zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex or a leaf. An edge e in a graph G is called a pendant edge if it is incident with a pendant vertex. Note: in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum. This result is called the handshaking lemma.

A graph $G = (V, E)$ is a simple graph if G has no edges that are self-loops and the set $E(G)$ consists of distinct unordered pairs of distinct elements of $V(G)$. Thus every simple graph is a graph, but not every graph is a simple graph, we will assume that every graph we discuss in these notes is a simple graph and we will use the term graph to mean simple graph. When a particular result holds in a more general setting, we will state it explicitly.

A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is a proper subgraph of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A spanning subgraph of G is a subgraph H of G with $V(H) = V(G)$.

Much of graph theory involves 'walks' of various kinds. A walk is a sequence of vertices and edges $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_k$, in which each edge $e_i = v_{i-1}v_i$. This walk goes from v_0 to v_k or "connect" v_0 with v_k, the length of walk is the number of its edges, and if $v_0 = v_k$ the walk is closed. The important types of walk are: the path is a walk, the trail is a walk in which no edge is repeated, and the cycle is a non trivial closed trail in which no vertex is repeated. Usually the path with n vertices is denoted by P_n and the cycle with n vertices by C_n, the least cycle (when $n = 3$) is called triangle. The path P_6 and cycle C_6 are shown in Figure 1.1.

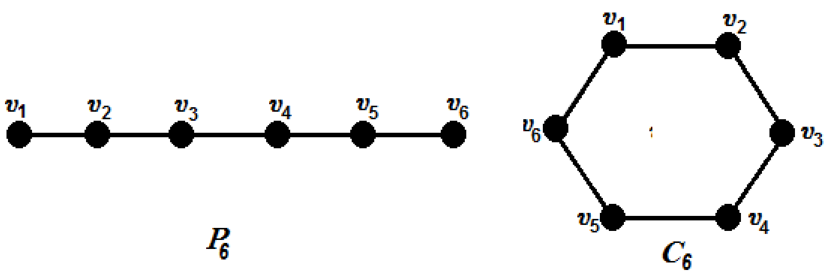

Figure 1.1: The path and the cycle.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \longrightarrow V_2$ that preserves the adjacency, i.e., $uv \in E_1$, if and only if, $f(u)f(v) \in E_2$. The function f is then called an isomorphism between G_1 and G_2. If G_1 and G_2 are isomorphic,then we write $G_1 \cong G_2$, the two graphs G_1 and G_2 shown in Figure 1.2 are isomorphic.

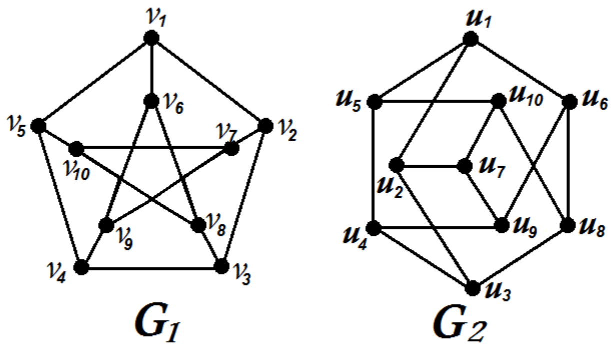

Figure 1.2: A two isomorphic graphs.

Clearly "if two graphs are isomorphic then they have, same number of vertices, same number of edges and umber of vertices" having same degree is equal.

A graph is said to be connected if for every pair of distinct vertices u, v there is a $u, \cdots, v$ *path* joining them. A graph that is not connected is called disconnected, a maximal connected subgraph of a disconnected graph is called a component of the graph and every connected graph has exactly one component, in other words, a graph is connected if it cannot be expressed as the union of two graphs, and disconnected otherwise. Each one of the two graphs G_1 and G_2 shown in Figure 1.2 is connected.

1.2.1 Some Types of Graphs

This subsection presents the definitions for some types of graphs that we may remember or study in this work.

A Null graph is a graph with n vertices and has no edge. A graph in which all the vertices have equal degree is called a regular graph. If for every vertex v of graph G, $d(v) = k$ for some $k \in \mathbb{N}$, then G is $k-$regular graph [14]. The null graph is $0 - regular$ graph and every cycle C_n is a $2-$regular graph with n vertices.

A complete graph $G = (V, E)$ on n vertices has n vertices and for each pair of vertices u, v; $uv \in E(G)$. That means in the complete graph with n vertices every two of which are adjacent, then: $|E\,(G)| = \dfrac{1}{2}n(n-1)$. A complete graph on n vertices is denoted by K_n . Note that K_n is $(n-1) - regular$ and the null graph with one vertex is K_1.

A graph $G = (V, E)$ is said to be bipartite if the vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that for every edge $e_i = v_i v_j \in E,\ v_i \in V_1\ and\ v_j \in V_2$. Figure 1.3 shows the bipartite graph. Notice the vertices set of bipartite graph contain at least two vertices and at least one edge.

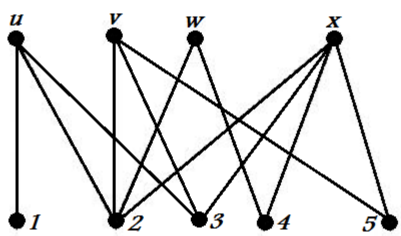

Figure 1.3: The bipartite graph.

We can also define $n-partite$ graphs as: A graph G = (V,E) is called $n-partite$ graph if the vertex set V can be partitioned into n nonempty sets $V_1, V_2, ..., V_n$ such that every edge of G joins the vertices from different subsets. It is often called a *multipartite* graph.

The complete bipartite graph $K_{m,n}$ is the bipartite graph whose vertex set is partitioned into two non-empty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$, and any vertex in V_1 is adjacent to each vertex in V_2, and any two distinct vertices in V_i are not adjacent to each other. The number of edges in complete bipartite graph $K_{m,n}$ is mn. If $m = n$, then $K_{n,n}$ is $n - regular$. When $m = 1$ then $K_{1,n}$ is called the star graph.

An $n-partite$ graph G is called complete $n-partite$ if for each $i \neq j$, each vertex of the subset V_i is adjacent to every vertex of the subset V_j. A complete $n - partite$ graph with n partitions of vertex set is denoted by $K_{m_1,m_2,...,m_n}$.

A graph is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no any two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar graph. A simple planar graph is called maximal planar if no edge can be added without destroying its planarity [6].

A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same "face". An outerplanar graph is maximal if no edge can be added without losing outerplanarity [6].

A graph which does not contain any cycle is known an acyclic graph and if it includes exactly one cycle it is called a unicyclic graph [4].

A forest is a graph that contains no cycles, and a connected forest is a tree [48]. For a tree T of order n, T is connected, has no cycles, and has $n-1$ edges [48]. A star graph with n vertices is a tree with one vertex having degree $n-1$ and other $n-1$ vertices having degree 1 and denoted by S_{n-1}. There are many types of trees we will discuss some of them in chapter five.

A cycle passing through all the vertices of a graph is called Hamiltonian graphs. A graph containing a Hamiltonian cycle is called Hamiltonian graphs. A path passing through all the vertices of a graph is called Hamiltonian path and a graph containing Hamiltonian path is said to be semi-Hamiltonian.

Define the general form of Petersen graph as: the graph $P(k,m) = (V,E)$ where $V = \{u_i, v_i : i = 1, 2, ..., k-1\}$ and $E = \{u_i u_{i+l}, v_i v_{i+m}, u_i v_i : i = 0, 1, ..., k-1\}$ where addition is *modulo* k and $m < \frac{1}{2}\ k$ [15]. The Petersen graph $P(5,2)$ which shown as G_1 in Figure 1.2 consider the stander Petersen graph.

For a graph $G = (V,E)$ define the line graph, denoted by $L(G)$, as the graph with vertices consisting of the edges of G, that is $V(L(G)) = E(G)$, and where $e, \acute{e} \in V(L(G))$ are adjacent in $L(G)$ if, and only if, they are adjacent in G [2].

Let $G = (V,E)$ be a graph, the graph complement of G is the graph $G^c = (V, E^c)$ so that: $E^c = \{uv : u, v \in V and\ u \neq v\ and\ uv \notin\ E\}$ [2]. A graph is said to be self complement if $G \cong G^c$. The complement of complete graph K_n is the null graph with n vertices.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. It is said that G_1 and G_2 are disjoint if they have no vertex in common and edge disjoint if they have no edge in common.

1.2.2 Operations on Graphs

There are several ways to get new graphs from old. Briefly described some of these operations in this subsection:

The disconnected graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$ [14]. The union of k graphs isomorphic to G is denoted by kG [4]. The one point union of t cycles, each of length n is denoted by $C(t)$ n is called the friendship graph.

If $V1 \cap V2 = V$, then the graph $G = (V, E)$, where $V = V1 \cap V2$ and $E = E1 \cap E2$ is called the intersection of G_1 and G_2 and is denoted by $G_1 \cap G_2$.

If G_1 and G_2 are disjoint graphs, then the join of G_1 and G_2 is denoted by $G_1 + G_2$ and is defined as $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$

The Cartesian product $G \times H$ has vertex-set $V(G) \times V(H)$, and (v_i, w_j) is adjacent to (v_h, w_k) if either v_i is adjacent to v_h in G and $w_j = w_k$, or $v_i = v_h$ and w_j is adjacent to w_k in H. The ladder graph L_n is the Cartesian product of the paths P_n *and* P_2, i.e. $L_n = P_n \Box P_2$. Figure 1.4 shows some operation on graphs.

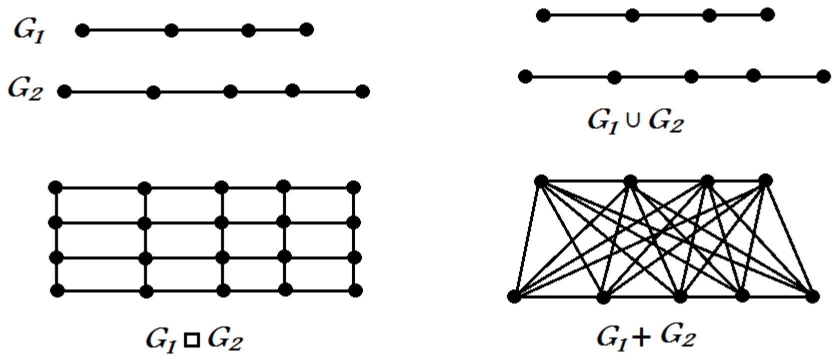

Figure 1.4: Some operations on graphs

The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (of V_1 vertices) and V_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2.

Chapter 2

On Difference Cordial Graphs

In this chapter we introduce some results in difference cordial graphs and the difference cordial labeling for some families of graphs as: ladder, triangular ladder, grid, step ladder and two sided step ladder graph. Also we discussed some families of graphs which may be difference cordial or not, such as diagonal ladder graphs and some types of one-point union of graphs.

2.1 Introduction

In this chapter we will deal with finite, simple and undirected graphs. By the expression $G = (V, E)$ we mean a simple undirected graph with vertex set V, $|V|$ is called the order of graph and edge set E, $|E|$ is called its size. Graph labeling connects many branches of mathematics and is considered one of important blocks of graph theory, for more details see [14].

Cordial labeling was first introduced in 1987 by Cahit [8], then there was a major effort in this area made this topic growing steadily and widely, see [12].

In [23] Ponraj, Shathish Naraynan and Kala introduce the notions of difference cordial labeling for finite undirected and simple graph, as in the following definition:

Definition 1. *Let $G = (V, E)$ be a (p, q) graph, and f be a map from $V(G)$ to $1, 2, ..., p$. For each edge uv assign the label $|f(v) - f(u)|$, f is called a difference cordial labeling if f is one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ denotes the number of edges labeled with 1 while $e_f(0)$ denotes the number of edges not labeled with 1. A graph with a difference cordial labeling is called a difference cordial graph [23].*

Ponraj et al. show every graph is a subgraph of a difference cordial graph and any $r-$regular graph with $r \geq 4$ is not difference cordial graph, every path and cycle are difference cordial graphs, the star graph $K_{1,n}$ is difference cordial if and only if $n \leq 5$, the graph K_n is difference cordial only when $n \leq 4$ while the bipartite graph $K_{m,n}$ is not difference cordial if $m \geq 4$ and $n \geq 4$, the bistar $B_{m,n}$ is not difference cordial when $m+n \geq 9$ but the wheel W_n, the fan F_n, the gear G_n, the helm H_n and all webs are difference cordial graphs for all n [23].

In [24] the authors investigated the difference cordial labeling behavior of $G \odot P_n$, $G \odot mK_1\,(m = 1, 2, 3)$ where G is either unicyclic or a tree and $G_1 \odot G_2$ are some more standard graphs. Some graphs obtained from triangular snake and quadrilateral snake were investigated with respect to the difference cordial labeling behavior. Also the behavior of subdivision of some snake graphs is investigated in [24].

Proposition 2.1. *If G is a (p, q) difference cordial graph, then $q \leq 2p - 1$ [23].*

Definition 2. *The number $\delta(G) = \min\{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max\{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, the number $d(G) = \frac{1}{|V|} \sum\limits_{v \in V} d(v)$ is the average degree of the vertices in the graph G [10].*

Definition 3. *A fan graph is obtained by joining all vertices of a path P_n to a further vertex, called the center. Thus F_n contains $n + 1$ vertices say $c, v_1, v_2, v_3, ..., v_n$ and $2n - 1$ edges, say $cv_i, 1 \leq i \leq n$, and, $v_i v_{i+1}, 1 \leq i \leq n - 1$.*

Notation 2.2. *The maximum number of edges labeled 1, that is related with a specific vertex, equals 2.*

2.2 Main Results

Proposition 2.3. *The graph $G(p, q)$ is not difference cordial graph if $\delta(G) \geq 4$.*

Proof. Let $G(p, q)$ be any graph with $\delta(G) \geq 4$; then, the minimum value of q is $2p$; but $2p > 2p - 1$, this contradicts proposition 2.1. □

Proposition 2.4. *The graph $G(p,q)$ is not difference cordial if $d(G) \geq 4$.*

Proof. Let $G(p,q)$ be any graph with $d(G) \geq 4$; then the value of q is more than or equal to $2p$, but $2p \nleq 2p-1$, which is contradicts Proposition 2.1. □

Remark 2.5. *The value of $e_f(0)$ is not exceeding p in any difference cordial graph $G(p,q)$.*

Proof. Direct consequence of Proposition 2.1. □

Proposition 2.6. *Let $G(p,q)$ be a graph with two vertices of degree $(p-1)$, then G is not difference cordial for all $p \geq 8$.*

Proof. Let $G(p,q)$ be a graph with p vertices, $p \geq 8$ and has two vertices v_i, v_j of degree $(p-1)$ then there are $2p-3$ different edges incident with them, If there are more than two additional edges then G is not difference cordial since $q \nleq 2p-1$. If there are only two additional edges then $q = 2p-1$, then we have two cases:

Case 1: the edge connecting v_i and v_j is labeled 0, then there are at most 6 edges are labeled 1: two passing through v_i, two are passing through v_j and the two additional edges.
in this case

$$|2p-7-6| = |2p-13| \geq 2 \;\; where \;\; p \geq 8$$

i.e., G is not difference cordial.

Case 2: the edge connecting v_i and v_j are labeled 1, then there are at most 5 edges labeled 1: one passing through v_i and v_j, two edges are: one is incident with v_i and other is incident with v_j and the two additional edges. In this case

$$|2p-6-5| = |2p-11| \geq 2 \; where \;\; p \geq 7$$

i.e., G is not difference cordial.

In case there is one additional edge, other than those incident with v_i,v_j, similar argument is used. □

Example 2.7. $\deg(V_8) = 7$, $\deg(V_7) = 7$

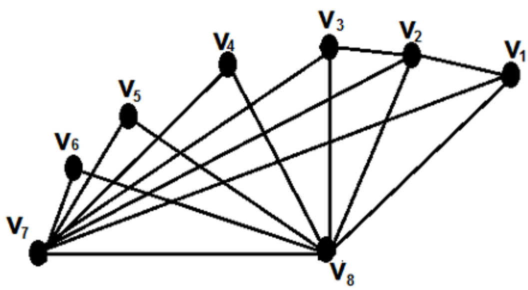

Figure 2.1: The graph $G = (8, 15)$

notice: G cannot be a difference cordial graph.

Proposition 2.8. *Let $G(p, q)$ be any graph with two vertices of degrees $(p-1)$ and $(p-2)$; then G is not a difference cordial graph for all $p \geq 9$.*

Proof. Similar to the proof of Proposition 2.6. □

Example 2.9. *In Figure 2.2*

$\deg(V_8) = 7$, $\deg(V_9) = 8$, *G cannot be a difference cordial graph.*

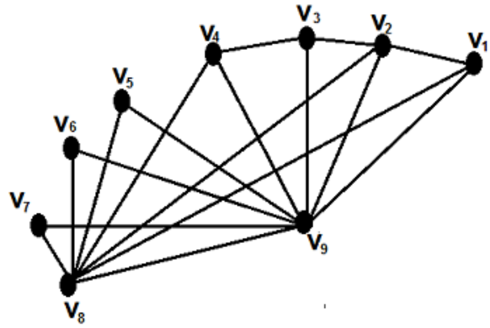

Figure 2.2: The graph $G = (9, 17)$

In [25] theorem 2.14, R. Ponraj, S. Sathish Narayanan and R. Kala state that, Let G be a (p, q) difference cordial graph with $k(k > 1)$ vertices of degree $p - 1$. Then $p \leq 7$. However:

Corollary 2.10. *The graph $G(p, q)$ is not a difference cordial graph if there exist three vertices of degree $(p - 1)$ for all $p \geq 6$.*

Proof. Let $G(p, q)$ be a graph with three of its vertices of degree $p - 1$ then there exist at least $3p - 6$ edges in the graph, by proposition 2.1 if the graph is a difference cordial graph then

$$3p - 6 \leq 2p - 1$$

A contradiction when $p \geq 6$. □

Example 2.11. *In Figure 2.3*

$12 \not\leq 2 * 6 - 1$ *and G cannot be a difference cordial graph.*

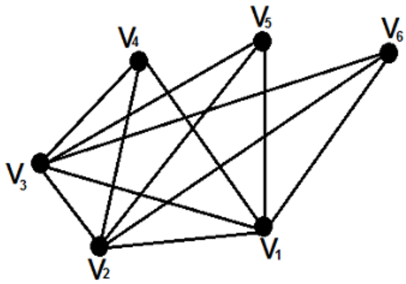

Figure 2.3: The graph $G = (6, 12)$

Proposition 2.12. *Let G be a (p, q) graph with one vertex of degree $(p-1)$ then G is not a difference cordial if there exists a set of non adjacent vertices S with $\sum_{v_i \in S}(\deg(v_i) - 3) \geq 4$.*

Proof. Let G be a (p, q) graph with p vertices and have a vertex v_k of degree $p-1$ and there exists a set of non adjacent vertices S with $\sum_{v_i \in S}(\deg(v_i) - 3) \geq 4$. Then there are at least $p-3$ edges passing through v_k are labeled 0, hence $e_f(0) \geq p - 3 + 4 = p + 1$, i.e., G is not a difference cordial graph. □

Example 2.13. *In the Figure 2.4*

$n = 17$, $q = 32$, $\deg(v) = 16$

$S = \{v_1, v_3, v_5, v_7\}$ *then*

$\sum_{v_i \in S}(\deg(v_i) - 3) = 1 + 1 + 1 + 1 = 4$

there are at least $4 + 14 = 18$ edges labeled 0, then the graph is not difference cordial.

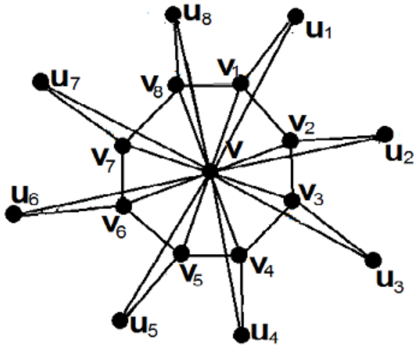

Figure 2.4: The flower graph Fl_8

Proposition 2.14. *Let G be a (p, q) graph then G is not difference cordial graph if there exists a set of non adjacent vertices S with $\sum_{v \in S}(\deg(v) - 2) = p + 1$.*

Proof. Let S be a set of non adjacent vertices with $\sum_{v_i \in S}(\deg(v_i) - 2) = p + 1$. Since the maximum number of edges labeled 1 that are incident with a specific vertex equals 2, then the number of edges labeled 0 that are incident with vertices of S are at least $\sum_{v_i \in S}(\deg(v_i) - 2)$ this means the minimum value for $e_f(0)$ in the graph G is $p + 1$, therefor the graph cannot be difference cordial. □

Proposition 2.15. *The complement graph of a difference cordial graph is not difference cordial when the number of its vertices is more than eight.*

Proof. Let G be a (p, q) difference cordial graph with $p \geq 9$, then by Proposition 2.6 :

$$q \leq 2p - 1 \tag{2.1}$$

G^c, the complement of graph G contains $\frac{1}{2}p(p-1)-q$ edges and p vertices, let G^c be difference cordial then

$$\frac{1}{2}p(p-1)-q \leq 2p-1 \tag{2.2}$$

by adding equation 2.1 and equation 2.2 we get

$$\frac{1}{2}p(p-1) \leq 4p-2p^2-9p \leq -4$$

A contradiction for all $p \geq 9$ □

2.3 Difference cordial labeling for Some graphs

In This section we will discuss the ability of applying difference cordial labeling for some graphs and the functions which make it difference cordial graphs.

The Proposition 2.1 consider necessary condition for difference cordial labeling but it is not sufficient.

2.3.1 Ladder graphs L_n

The ladder graph is a planner undirected graph denoted by L_n with $2n$ vertices and $3n-2$ edges [14]. The ladder graph L_n can be expressed as $L_n \cong P_n \times P_2$, the Figure 2.5 show the ladder graph L_n

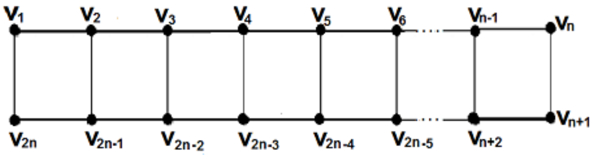

Figure 2.5: The Ladder Graph L_n

Proposition 2.16. *Every ladder graph L_n is difference cordial for all n.*

Proof. Let L_n be a ladder graph, then it has $2n$ vertices and $3n-2$ edges. Let the vertices be $v_1, v_2, ..., v_{2n}$ such that $v_n v_{n+1}$ is an edge in this graph. Define the mapping $f : L_n \longrightarrow \{1, 2, ..., 2n\}$ by:

$$f(v_i) = \left\{ \begin{array}{ll} i & if \;\; 1 \leq i \leq \left\lceil \frac{1}{2}|E| \right\rceil \\ 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2i & if \;\; \left\lceil \frac{1}{2}|E| \right\rceil < i \leq \left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil \\ 2(i-n) - 1 & if \;\; \left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil < i \leq 2n \; and \; n \; is \; odd \\ 2(i-n) & if \;\; \left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil < i \leq 2n \; and \; n \; is \; even \end{array} \right\}$$

From the first part of definition notice that there are $\left\lceil \frac{1}{2}|E| \right\rceil - 1$ of edges are labelled 1, in the second part we notice that:

$$\begin{aligned} |f(v_{i+1}) - f(v_i)| &= \left| 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2(i+1) - 3\left\lceil \frac{1}{2}|E| \right\rceil - \left\lceil \frac{1}{2}n \right\rceil - 2 + 2i \right| \\ &= 2 \end{aligned}$$

so

$$\begin{aligned}\left|f(v_i) - f(v_{2n-(i+1)})\right| &= \left|3\left\lceil \frac{1}{2}\left|E\right|\right\rceil + \left\lceil \frac{1}{2}n\right\rceil + 2 - 2i - 2n + (i+1)\right| \\ &= \left|3\left\lceil \frac{1}{2}\left|E\right|\right\rceil + \left\lceil \frac{1}{2}n\right\rceil + 3 - i - 2n\right| \\ &= \left|3\left\lceil \frac{1}{2}(3n-2)\right\rceil + \left\lceil \frac{1}{2}n\right\rceil + 3 - i - 2n\right| \\ &= |3n - i| > 1\end{aligned}$$

which means all these edges are labelled 0.

In the third part of definition we notice when n is even:

$$|f(v_{i+1}) - f(v_i)| = |2(i+1-n) - 2(i-n)| = 2$$

and

$$\begin{aligned}\left|f(v_i) - f(v_{2n-(i+1)})\right| &= |2(i-n) - 2n + (i+1)| = |3i - 4n| \\ &> \left|3(\left\lceil \frac{1}{2}\left|E\right|\right\rceil + \left\lceil \frac{1}{4}n\right\rceil) - 4i\right| \\ &> \left|3(\left\lceil \frac{1}{2}(3n-2)\right\rceil + \left\lceil \frac{1}{4}n\right\rceil) - 4n\right| \\ &> \left|\frac{1}{2}n + 3\left\lceil \frac{1}{4}n\right\rceil - 3\right| \\ &> \left\{\begin{array}{lll} |5m-3| & if & n = 4m \\ & & \\ |5m+1| & if & n = 4m \end{array}\right\} \\ &> 2\end{aligned}$$

this means all the edges $v_i v_{2n-(i+1)}$ in this third part are labeled 0.

But if n is an even number then the number of the total edges of the ladder L_n is even and thus there must exist additional edge labeled 1, which we may get it from the label of the last vertex in part two and the first label in part three.

Notice that if $i = \left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil$ then

$$f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil}) = 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) \tag{2.3}$$

and if $i = \left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1$, then

$$f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}) = 2\left((\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1) - n\right) \tag{2.4}$$

by subtracting (2.4) from (2.3) we get:

$$\begin{aligned}
&f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil}) - f(v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}) \\
&= 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) - 2\left((\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1) - n\right) \\
&= -\left\lceil \frac{1}{2}|E| \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = -\left\lceil \frac{1}{2}(3n-2) \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil \\
&= \frac{-3}{2}n + 1 + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = n + 1 - 4\left\lceil \frac{1}{4}n \right\rceil = \left\{ \begin{array}{cc} 1 & if\ n = 4m \\ -1 & if\ n = 4m+2 \end{array} \right\}
\end{aligned}$$

thus the edge $v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil} v_{\lceil \frac{1}{2}|E| \rceil + \lceil \frac{1}{4}n \rceil + 1}$ is labelled 1, then the graph is difference cordial.

Now if n is an odd number then $|E|$ is an odd number and then from the first part we get $\left\lfloor \frac{1}{2}|E| \right\rfloor$ edges are labeled 1 and all other edges in the second and third part are labelled 0, similarly when n is even, and

$$f(v_{\lceil\frac{1}{2}|E|\rceil+\lceil\frac{1}{4}n\rceil}) - f(v_{\lceil\frac{1}{2}|E|\rceil+\lceil\frac{1}{4}n\rceil+1})$$
$$= 3\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{2}n\right\rceil + 2 - 2\left(\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{4}n\right\rceil\right) - 2\left((\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{4}n\right\rceil + 1) - n\right) + 1$$
$$= -\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{2}n\right\rceil - 4\left\lceil\frac{1}{4}n\right\rceil + 2n - 1$$
$$= -\left\lceil\frac{1}{2}(3n-2)\right\rceil + \left\lceil\frac{1}{2}n\right\rceil - 4\left\lceil\frac{1}{4}n\right\rceil + 2n - 1$$

$$= \left\{\begin{array}{ll} -\left\lceil\frac{1}{2}(3(4m+1)-2)\right\rceil + \left\lceil\frac{1}{2}(4m+1)\right\rceil & \\ \quad -4\left\lceil\frac{1}{4}(4m+1)\right\rceil + 2(4m+1) - 1 & if n = 4m+1 \\ & \\ -\left\lceil\frac{1}{2}(3(4m+3)-2)\right\rceil + \left\lceil\frac{1}{2}(4m+3)\right\rceil & \\ \quad -4\left\lceil\frac{1}{4}(4m+3)\right\rceil + 2(4m+3) - 1 & if n = 4m+3 \end{array}\right\}$$
$$= \left\{\begin{array}{lll} 0 & if & n = 4m+1 \\ 1 & if & n = 4m+3 \end{array}\right\}$$

then

$$e_f(1) = e_f(0) \qquad if\ n\ is\ even$$
$$e_f(1) = e_f(0) - 1 \qquad if\ n\ is\ odd\ \&\ n = 4m+1$$
$$e_f(1) = e_f(0) + 1 \qquad if\ n\ is\ odd\ \&\ n = 4m+3$$

Hence G is difference cordial. □

Example 2.17. *Consider the graph* L_{10}

$n = 10, |E| = 28, \ \left\lceil \frac{1}{2}|E| \right\rceil = 14, \ \left\lceil \frac{1}{2}n \right\rceil = 5, \left\lceil \frac{1}{4}n \right\rceil = 3$ *then:*

$$f(v_i) = \begin{cases} i & if \quad 1 \leq i \leq 14 \\ 49 - 2i & if \quad 14 < i \leq 17 \\ 2(i-10) & if \quad 17 < i \leq 20 \end{cases}$$

$f(v_1) = 1, \ f(v_2) = 2, \ \cdots, \ f(v_{14}) = 14, \ f(v_{15}) = 19, \ f(v_{16}) = 17, f(v_{17}) = 15, \ f(v_{18}) = 16, f(v_{19}) = 18, f(v_{20}) = 20.$

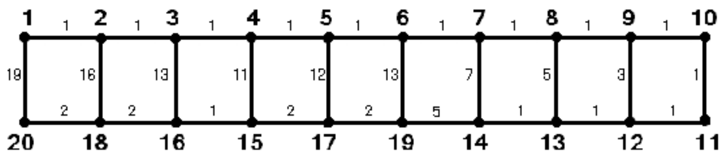

Figure 2.6: Ladder Graph L_{10}

$e_f(0) = 14, \quad e_f(1) = 14$

Example 2.18. *Consider the graph* L_{11}

$n = 11, |E| = 31, \left\lceil \frac{1}{2}|E| \right\rceil = 16, \left\lceil \frac{1}{2}n \right\rceil = 6, \left\lceil \frac{1}{4}n \right\rceil = 3,$

then

$$f(v_i) = \begin{cases} i & if \quad 1 \leq i \leq 16 \\ 56 - 2i & if \quad 16 < i \leq 19 \\ 2(i-n) - 1 & if \quad 19 < i \leq 22 \end{cases}$$

$f(v_1) = 1, \ f(v_2) = 2, \ \cdots, \ f(v_{16}) = 16, \ f(v_{17}) = 22, \ f(v_{18}) = 20, \ f(v_{19}) = 18, \ f(v_{20}) = 17, \ f(v_{21}) = 19, \ f(v_{22}) = 21.$

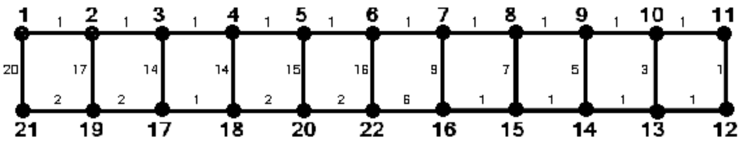

Figure 2.7: Ladder Graph L_{11}

$e_f(0) = 15,\ e_f(1) = 16$

2.3.2 Triangular ladder graph TL_n

A triangular ladder $TL_n, n \geq 2$, is a graph obtained from the ladder L_n $= P_n \times P_2$ by adding the edges $u_i v_{i+1} for 1 \leq i \leq n-1$, such graph has $2n$ vertices with $4n-3$ edges, the triangular ladder graph TL_n is shown in the Figure 2.8.

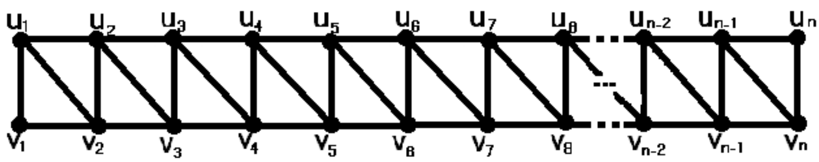

Figure 2.8: Triangle Ladder Graph TL_n

Proposition 2.19. *The triangular ladder graphs $TL_n, n \geq 2$ are difference cordial graph for all n.*

Proof. Let $G = TL_n, n \geq 2$ be a triangular ladder graph, then $G = (2n, 4n-3)$. Define the function

$$f(v_i) = 2i - 1 \; and \; f(u_i) = 2i, \quad 1 \leq i \leq n \tag{2.5}$$

It is clear that $e_f(1) = 2n - 1$

hence $e_f(0) = (4n-3) - (2n-1) = 2n-2$, then $|e_f(0) - e_f(0)| = 1$, thus $G = TL_n, n \geq 2$ are difference cordial graphs for all n. □

Example 2.20. *Consider the graphs TL_6 and TL_7*

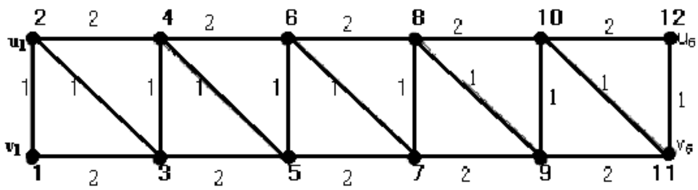

Figure 2.9: A difference cordial labeling for TL_6

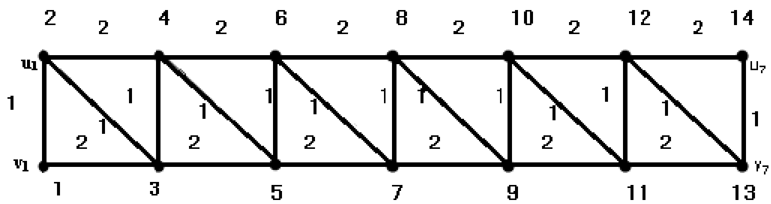

Figure 2.10: A difference cordial labeling for TL_7

2.3.3 The Grid graph $P_m \times P_n$

In this subsection we will investigate the difference cordial labeling for every grid graph of the form $P_m \times P_n$ for all m, n. Let the vertices of the grid graph be arranged as a sequence in certain order as in the Figure 2.11

This kind of graphs contains mn vertices and $2mn - (m+n)$ edges.

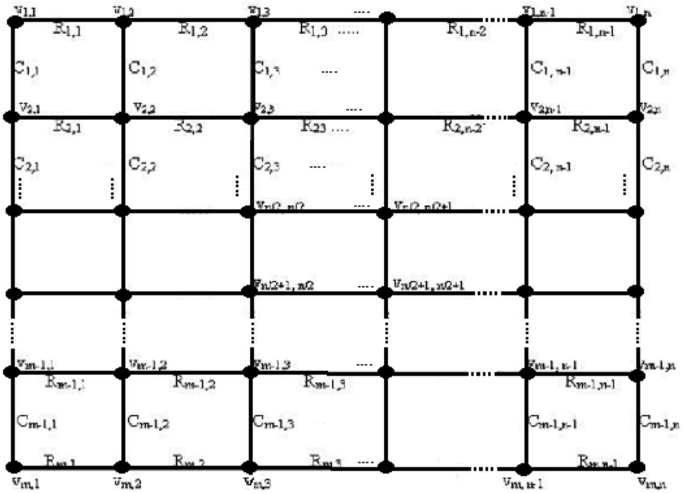

Figure 2.11: The grid graph $P_m \times P_n$

Proposition 2.21. *Every grid graph is $P_m \times P_n$ is a difference cordial graph for all integers $m, n > 1$.*

Proof. Let G be a graph $P_m \times P_n$ then $G = (mn, 2mn - (m+n))$
Case 1: If $m = n$

then $|V| = n^2$ and $|E| = 2(n^2 - n)$, define the function f for labeling vertices of G by:

$$f(v_{ij}) = (i-1)\,n + j$$

in each row of the grid graph there exist $n-1$ edges labelled 1 this leads to $e_f\,(1) = n(n-1)$ and the number of edges labelled 0 is equal to:

$$2n(n-1) - n(n-1) = n(n-1),$$

thus G is a difference cordial graph.

Case 2: If $|m-n|=1$, then $|V|=mn$ and $|E|=2mn-(m+n)$.
Let $n=m+1$ then $|E|=2m^2-1$.
Now using the same functions in **Case 1** we will get

$$e_f(1)=m(n-1)=m(m+1-1)=m^2$$

and

$$e_f(0)=(m-1)(m+1)=m^2-1$$

which means the graph is a difference cordial graph. Similarly if $m=n+1$

Case 3: If $|m-n|\geq 2$.
Let $n>m$ and let $k=\left\lceil\frac{1}{2}(n-m)\right\rceil$ we define the mapping:

$$f(v_{ij})=\begin{Bmatrix} (j-1)\,m+i & if\;\; 1\leq j\leq k \\ \\ k\,(m-1)+n\,(i-1)+j & if\;\; j=k+1,...,n \end{Bmatrix}$$

It follows that:

$$\begin{aligned} e_f\,(1) &= k(m-1)+m(n-k-1) \\ &= mn-(m+k) \end{aligned}$$

and

$$\begin{aligned} e_f(0) &= 2mn - (m+n) - mn + (m+k) \\ &= mn - (n-k) \end{aligned}$$

so

$$\begin{aligned} |e_f(0) - e_f(1)| &= |mn - n + k - mn + m + k| \\ &= |-n + 2k + m| \\ &= \left\{ \begin{array}{ll} 0 & if\ n - mis\ \ even \\ 1 & if\ \ n - m\ \ is\ odd \end{array} \right\} \end{aligned}$$

Similarly if $m > n$ we apply the same mapping but replacing i by j and m by n, i.e.:

$k = \left\lceil \frac{1}{2}(m-n) \right\rceil$ and:

$$f(v_{ij}) = \left\{ \begin{array}{ll} (i-1)\,n + j & if\ \ 1 \leq i \leq k \\ & \\ k\,(n-1) + m\,(j-1) + i & if\ \ i = k+1, ..., m \end{array} \right\}$$

Hence the grid graph $P_m \times P_n$ is a difference cordial graph for all m, n. □

Example 2.22. *Let* $P_m \times P_n = P_4 \times P_3$

$n = 3,\ m = 4,\quad |V| = 12,\quad |E| = 17$

$f(v_{ij}) = 3(i-1) + j$

$e_f(1) = 8, e_f(0) = 9.$

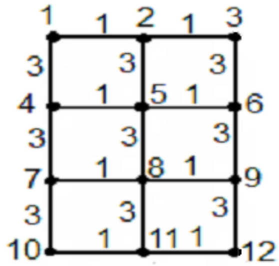

Figure 2.12: A difference cordial labeling for grid graph $P_4 \times P_3$

Example 2.23. *Let* $P_m \times P_n = P_5 \times P_8$

$n = 8,\ m = 5,\ |V| = 40, |E| = 67,\ k = 2,$ *then*

$$f(v_{ij}) = \begin{cases} 5(j-1) + j & 1 \le j \le 2 \\ 2(5-i) + 8(i-1) + j & j > 2 \end{cases}$$

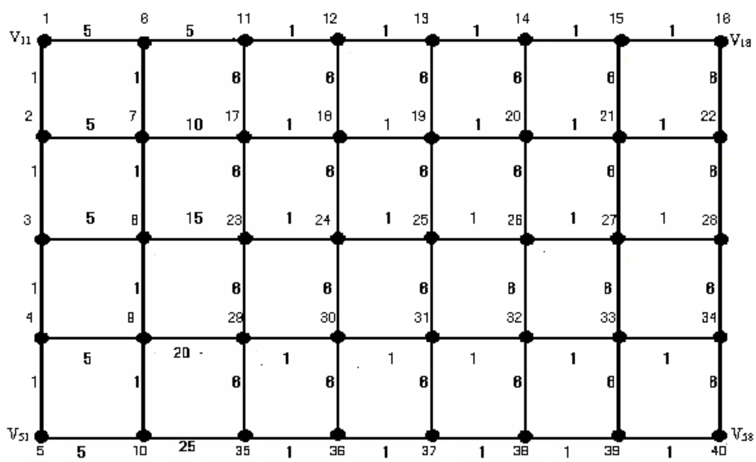

Figure 2.13: A difference cordial labeling for grid graph $P_5 \times P_8$

$e_f(0) = 34,\ e_f(1) = 33$

2.3.4 Step ladder graph $S(T_n)$:

Definition 4. *Let P_n be a path on n vertices denoted by $(1,1),(1,2),...,(1,n)$ and $n-1$ edges denoted by $e_1,e_2,...,e_{n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i, $i=1,2,\dots,n-1$ we erect a ladder with $n-(i-1)$ steps including the edge e_i. The graph obtained is called a step ladder graph and is denoted by $S(T_n)$, where n denotes the number of vertices in the base.*

The Figure 4.2 shows the step ladder graph:

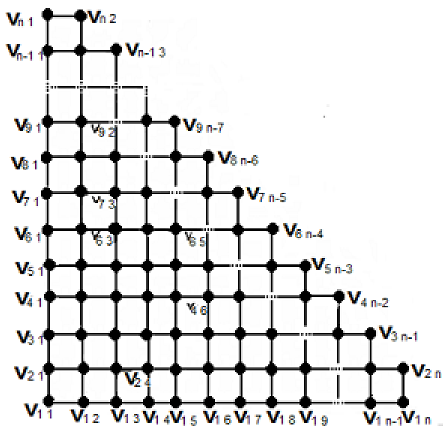

Figure 2.14: The step ladder graph $S(T_n)$.

The number of vertices and edges in the step ladder graph $S(T_n)$ are:

$$
\begin{aligned}
|V| &= 2+3+4+...+n+n \\
&= \frac{1}{2}n(n+1)+(n-1) \\
&= \frac{n^2+3n-2}{2}
\end{aligned}
$$

$$\begin{aligned}|E| &= 2(|V| - n)\\ &= n(n+1) - 2\end{aligned}$$

We notice for all step ladder graphs that $i + j \leq n + 2$

Proposition 2.24. *Every step ladder graph $S(T_n)$ is a difference cordial graph for all n.*

Proof. Let $S(T_n)$ be a step ladder graph then

$$|E| = n(n+1) - 2 = n^2 + n - 2$$

Define the function:
$f : V(S(T_n)) \longrightarrow \{1, 2, ..., \frac{1}{2}n(n+1) + (n-1)\}$ by:

$$f(v_{ij}) = \left\{ \begin{array}{ll} j + (i-1)n & 1 \leq i \leq 3 \\ \\ j + (i-1)n - \frac{1}{2}(i-3)(i-2) & i \geq 4 \end{array} \right\}$$

$$\begin{aligned}e_f(1) &= (3n-4) + (n-3) + (n-4) + (n-5) + ... + 3 + 2 + 1\\ &= (n-1) + (n-1) + (n-2) + (n-3) + ... + 2 + 1\\ &= (n-1) + \frac{1}{2}n(n-1)\\ &= \frac{1}{2}(n^2 + n - 2),\end{aligned}$$

then $e_f(1) = \frac{1}{2}|E|$ which means $|e_f(1) - e_f(0)| = 0$

Therefor $S(T_n)$ is a difference cordial graph for all n □

Example 2.25. *In the following figure the difference cordial labeling for $S(T_{12})$ graph.*

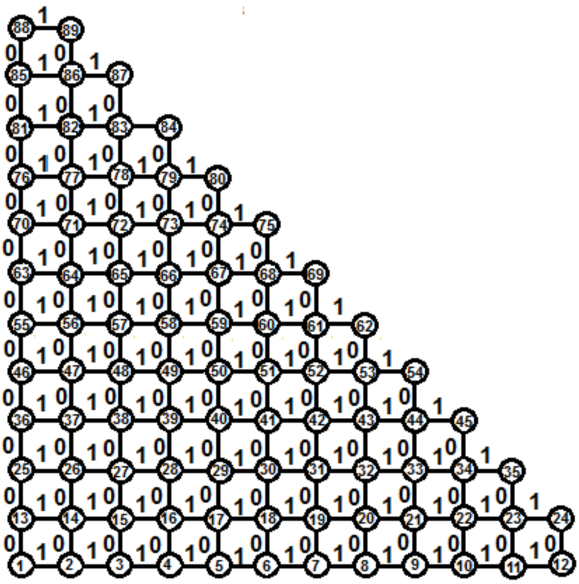

Figure 2.15: Difference cordial labeling for the step ladder graph $S(T_{12})$

2.3.5 Double Sided Step Ladder Graph $2S(T_{2n})$:

Definition 5. *Let P_{2n} be a path of length $2n-1$ with $2n$ vertices $(1,1),(1,2),...,(1,2n)$ with $(2n-1)$ edges, $e_1,e_2,...,e_{2n-1}$, where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i, for $i=1,2,...,n$, we erect a ladder with $(i+1)$ steps including the edge $e_i$and on each edge e_i, for $i=n+1,n+2,...,2n-1$, we erect a ladder with $2n+1-i$ steps including the edge e_i.*

The double sided step ladder graph $2S(T_{2n})$ *has vertices denoted by:* $(1,1),(1,2),...,(1,2n),(2,1),(2,2),...,(2,2n),(3,2),(3,3),...,$ $(3,2n-1),(4,3),(4,4),...,(4,2n-2),...,(n+1,n),(n+1,n+1)$. *In the ordered pair* (i,j), i *denotes the row number (counted from bottom to top) and* j *denotes the column number (from left to right) in which the vertex occurs.*

The figure 2.16 show the $2S(T_{10})$

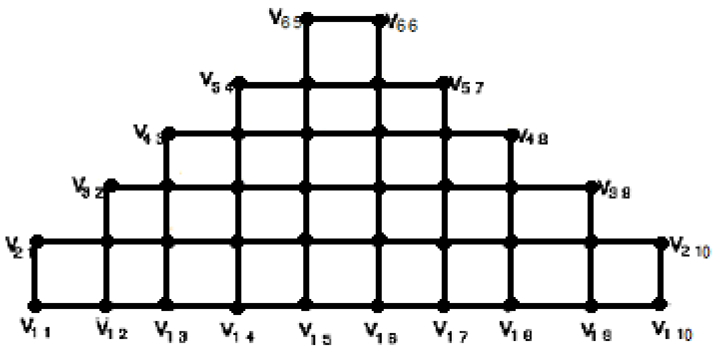

Figure 2.16: Double sided step ladder graph $2S(T_{10})$

Proposition 2.26. *The double sided step ladder graph* $2S(T_m)$ *is a difference cordial graph for all* m*, where* $m = 2n$ *denotes the number of vertices in the base.*

Proof. Let $G = (V,E)$ be the double sided step ladder graph $2S(T_m)$ where $m = 2n$, then

$$|V| = n^2 + 3n \qquad \textit{and} \qquad |E| = 2n^2 + 3n - 1$$

Define $f : V \longrightarrow 1, 2, ..., n^2 + 3n$ by:

$$f(v_{i,j}) = \begin{cases} u & if \quad i = 1 \ and \ j \leq \left\lceil \frac{1}{2}n \right\rceil \\ j + 2n(i-1) & if \ i = 1 \ and \ j \geq \left\lceil \frac{1}{2}n \right\rceil + 1 \\ j + 2n(i-1) & if \ i = 2 \\ j + 2n(i-1) - (i-1)^2 & if \ i = 3, 4, ..., n+1 \end{cases}$$

where

$$u = \begin{cases} 2j(\mathrm{mod} \left\lceil \frac{1}{2}n \right\rceil + 1) & if \quad n = 3 \quad or \quad n \equiv 0 \ (\mathrm{mod}\, 4) \\ 2j(\mathrm{mod} \left\lceil \frac{1}{2}n \right\rceil + 1) + \left\lfloor \frac{2j}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor & if \quad n \equiv 1, 2(\mathrm{mod}\, 4) \\ (2j-1)(\mathrm{mod} \left\lceil \frac{1}{2}n \right\rceil + 1) + 2 \left\lfloor \frac{2j}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor & if \quad n \equiv 3(\mathrm{mod}\, 4) \end{cases}$$

from the last three parts of the definition of f we will get $n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1$ edges give are labeled 1, while in the first part all edges are labeled 0 except when $\left\lceil \frac{1}{2}n \right\rceil \leq 4$ we will get an edge is labeled 1 since $1 \leq u \leq 3$

Case 1:

If $n = 2$ then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 7$ and $e_f(0) = 6$,
if $n = 3$ then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 13$ and $e_f(0) = 13$,

if $n = 4$ then $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 + 1 = 22$ and $e_f(0) = 21$ and if $n = 5$ then $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 + 1 = 32$ and $e_f(0) = 32$.

Case 2:

If $n \equiv 0 \pmod 4$ then $n = 4k$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k$, then $|E| = 2(4k)^2 + 3(4k) - 1 = 32k^2 + 12k - 1$

and

$$\begin{aligned} f(v_{1\lceil \frac{1}{2}n \rceil}) &= 2 \left\lceil \frac{1}{2}n \right\rceil \text{ (mod } (\left\lceil \frac{1}{2}n \right\rceil + 1)) \\ &= (2 * 2k) \text{ (mod } (2k+1)) = 4k \text{ (mod } (2k+1)) \\ &= 2k - 1 \end{aligned}$$

while $f(v_{1\lceil \frac{1}{2}n \rceil + 1}) = \lceil \frac{1}{2}n \rceil + 1 = 2k + 1$,

thus the label of the edge $v_{1\lceil \frac{1}{2}n \rceil} v_{1\lceil \frac{1}{2}n \rceil + 1}$ will be included in $e_f(0)$,

therefor $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 = 16k^2 + 8k - 2k - 1 = 16k^2 + 6k - 1$ and

$e_f(0) = |E| - e_f(1) = 32k^2 + 12k - 1 - 16k^2 - 6k + 1 = 16k^2 + 6k$

we get $|e_f(0) - e_f(1)| = 1$

Case 3:

If $n \equiv 1 \pmod 4$ then, $n = 4k + 1$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k + 1$ then

$|E| = 2(4k+1)^2 + 3(4k+1) - 1 = 32k^2 + 28k + 4$, and

$$\begin{aligned} f(v_{1\ \lceil \frac{1}{2}n \rceil}) &= 2(\left\lceil \frac{1}{2}n \right\rceil)(\text{mod}(\left\lceil \frac{1}{2}n \right\rceil + 1)) + \left\lfloor \frac{2\left\lceil \frac{1}{2}n \right\rceil}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor \\ &= 2(2k+1)(\text{mod } (2k+2)) + \left\lfloor \frac{2\lceil 2k+1 \rceil}{\lceil 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+2) \ (\text{mod } (2k+2)) + 1 \\ &= 2k+1 \end{aligned}$$

while $f(v_{1\ \lceil \frac{1}{2}n \rceil + 1}) = \lceil \frac{1}{2}n \rceil + 1 = 2k+2$,

thus the label of the edge $v_{1\ \lceil \frac{1}{2}n \rceil} v_{1\ \lceil \frac{1}{2}n \rceil + 1}$ will be included in $e_f(1)$,

therefor $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n \rceil - 1 + 1 = (4k+1)^2 + 2(4k+1) - 2k - 1 - 1 + 1 = 16k^2 + 14k + 2$ and $e_f(0) = |E| - e_f(1) = 32k^2 + 28k + 4 - 16k^2 - 14k - 2 = 16k^2 + 14k + 2$

we get $|e_f(0) - e_f(1)| = 0$

Case 4:

If $n \equiv 2 \pmod 4$ then $n = 4k+2$ for some positive integer number k and $\lceil \frac{1}{2}n \rceil = 2k+1$ then $|E| = 2(4k+2)^2 + 3(4k+2) - 1 = 32k^2 + 44k + 13$

and

$$\begin{aligned} f(v_{1\lceil \frac{1}{2}n \rceil}) &= (2\left\lceil \frac{1}{2}n \right\rceil)(\text{mod} \left\lceil \frac{1}{2}n \right\rceil + 1) + \left\lfloor \frac{2\left\lceil \frac{1}{2}n \right\rceil}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor \\ &= 2(2k+1)(\text{mod}(2k+2) + \left\lfloor \frac{2\lceil 2k+1 \rceil}{\lceil 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+2)(\text{mod}(2k+2)) + 1 \\ &= 2k+1 \end{aligned}$$

while $f(v_{1\ \lceil\frac{1}{2}n\rceil+1}) = \lceil\frac{1}{2}n\rceil + 1 = 2k+2$,
thus the label of the edge $v_{1\ \lceil\frac{1}{2}n\rceil}v_{1\lceil\frac{1}{2}n\rceil+1}$ will included in $e_f(1)$,
therefore

$$\begin{aligned} e_f(1) &= n^2 + 2n - \left\lceil\frac{1}{2}n\right\rceil - 1 + 1 \\ &= (4k+2)^2 + 2(4k+2) - 2k - 1 - 1 + 1 \\ &= 16k^2 + 22k + 7 \end{aligned}$$

and

$$\begin{aligned} e_f(0) &= |E| - e_f(1) \\ &= 32k^2 + 44k + 13 - 16k^2 - 22k - 7 \\ &= 16k^2 + 22k + 6 \end{aligned}$$

we get $|e_f(0) - e_f(1)| = 1$

Case 5:

If $n \equiv 3 (\bmod 4)$, $n = 4k+3$ for some positive integer number k and $\lceil\frac{1}{2}n\rceil = 2k+2$, then

$$|E| = 2(4k+3)^2 + 3(4k+3) - 1 = 32k^2 + 60k + 26$$

and

$$\begin{aligned} f(v_{1\ \lceil \frac{1}{2}n \rceil}) &= (2\left\lceil \frac{1}{2}n \right\rceil - 1) \pmod{(\left\lceil \frac{1}{2}n \right\rceil + 1)} + 2\left\lfloor \frac{2\left\lceil \frac{1}{2}n \right\rceil - 1}{\left\lceil \frac{1}{2}n \right\rceil + 1} \right\rfloor \\ &= (2(2k+2) - 1) \pmod{(2k+3)} + 2\left\lfloor \frac{2\lceil 2k+1 \rceil - 1}{\lceil 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+3) \pmod{(2k+3)} + 2 \\ &= 2k+2, \end{aligned}$$

while $f(v_{1\ \lceil \frac{1}{2}n \rceil + 1}) = \lceil \frac{1}{2}n \rceil + 1 = 2k+3$,
thus the label of the edge $v_{1\ \lceil \frac{1}{2}n \rceil} v_{1\ \lceil \frac{1}{2}n \rceil + 1}$ will be included in $e_f(1)$, therefor

$$\begin{aligned} e_f(1) &= n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 \\ &= (4k+3)^2 + 2(4k+3) - 2k - 2 - 1 + 1 \\ &= 16k^2 + 30k + 13 \end{aligned}$$

and

$$\begin{aligned} e_f(0) &= |E| - e_f(1) \\ &= 32k^2 + 60k + 26 - 16k^2 - 30k - 13 \\ &= 16k^2 + 30k + 13 \end{aligned}$$

we get $|e_f(0) - e_f(1)| = 0$.

From the cases 1, 2, 3, 4 and 5 we conclude that the double sided step ladder graph $2S(T_{2n})$ is a difference cordial graph for all integer number n □

We discuss here some types of graphs not always difference cordial such as diagonal ladder graph, diagonal grid graph and friendship graph.

Diagonal ladder graph is a ladder with additional edges u_iv_{i+1} and $u_{i+1}v_i$, denoted by DL_n, where n is half its vertices and the number of its edges is $5n-4$

Corollary 2.27. *The diagonal ladder graphs are difference cordial if $n \leq 3$.*

Proof. Let the graph G be the diagonal ladder graph DL_n with $2n$ vertices that means there are $5n-4$ edges in G, G is a difference cordial graph. Then by proposition 2.6 we get,

$$5n - 4 \leq 2(2n) - 1$$
$$n \leq 3$$

then the diagonal ladder graph is difference cordial when $n = 2$ or $n = 3$ □

Example 2.28 shows that DL_2 and DL_3 are difference cordial.

Example 2.28. *The following are labeling for the diagonal ladder graphs* DL_2, DL_3

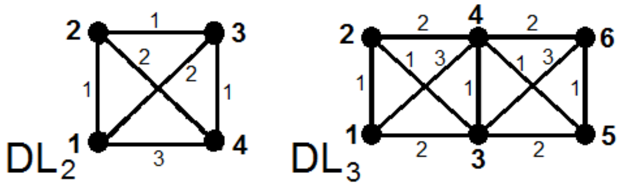

Figure 2.17: The difference cordial labelings for the diagonal ladder graphs DL_2 and DL_3.

The graph $P_m \times P_n$ with diagonal edges is called diagonal grid graph and denoted by $D(P_m \times P_n)$. It has mn vertices and $2(2mn+1) - 3(m+n)$ edges.

Remark 2.29. *Diagonal grid graph* $P_m \times P_n$ *are not difference cordial graphs for both* $m, n \geq 3$

Proof. Let $G = D\left(P_m \times P_n\right)$, from Proposition 2.1 if G is a difference cordial graph then $q \leq 2p - 1$. Let $m = n = 3$, then

$$\begin{aligned} q &= 2(2mn+1) - 3(m+n) \\ &= 2(2 \cdot 3 \cdot 3 + 1) - 3(3+3) \\ &= 20 \nleq 17 \end{aligned}$$

then $D\left(P_m \times P_n\right)$ cannot be a difference cordial graph for both $m, n \geq 3$. $\square$

This is consistent with corollary 2.27 since diagonal ladder graphs are diagonal grid graphs.

Another type of graphs will be discussed here named one-point union fan graph, where a graph G in which a vertex distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph, the Graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called a one-point union of the n copies of G.

Proposition 2.30. *The fan graph F_n is difference cordial for all n.[23]*

Proposition 2.31. *The one-point union $F_n^{(m)}$ of m copies of a fan F_n is difference cordial for all n and for $m \leq 5$.*

Proof. Let $G = F_n^{(m)}$, then $|V(G)| = mn + 1$ and $|E(G)| = m(2n - 1)$. These vertices are : the central vertex is denoted by v_{00} and the other vertices are denoted by $v_{i\ j}$, $1 \leq i \leq n$ and $1 \leq j \leq m$, as in the Figure 2.18.

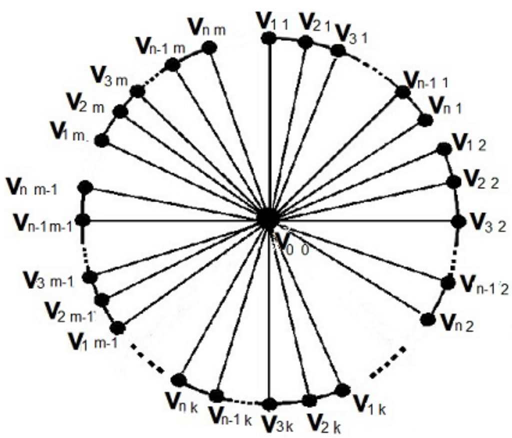

Figure 2.18: The graph $F_n^{(m)}$.

For each copy of a fan F_n there are $n-1$ edges labeled 1, therefore there are $m(n-1)+2$, edges are labeled 1 in $F_n^{(m)}$ where the central vertex is labeled $1 \pmod n$ but is neither 1 nor $mn+1$ then

$$e_f(0) = m(2n-1) - m(n-1) - 2 = mn - 2$$

Now

$$\begin{aligned}
&|e_f(0) - e_f(1)| \\
&= |mn - 2 - m(n-1) - 2| \\
&= |m - 4|
\end{aligned}$$

then $|e_f(0) - e_f(1)| \geq 2$ for all $m \geq 6$.

We define the mapping f for $m \leq 5$ and $n \in \mathbb{N}$ by

$$f(v_{0\ 0}) \equiv 1 (\operatorname{mod} n) \quad and \quad f(v_{0\ 0}) \neq 1, mn+1$$

and

$$f(v_{i\ j}) = \left\{ \begin{array}{lll} (j-1)n+i & if & (j-1)n+i < f(v_{0\ 0}) \\ \\ (j-1)n+i+1 & if & (j-1)n+i > f(v_{0\ 0}) \end{array} \right\}$$

for all $i, j;\ 1 \leq i \leq n,\ 1 \leq j \leq m.$ □

As a special case, the friendship graph denotes by $F_2^{(m)}$ consists of one vertex union with m copies of paths P_2 consisting of $2m+1$ vertices and $3m$ edges as shown in Figure 2.19

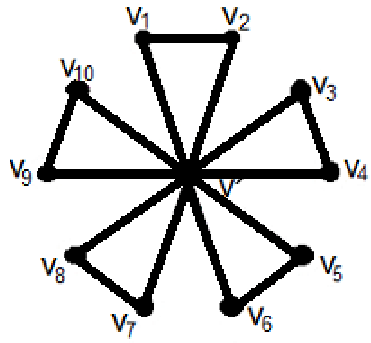

Figure 2.19: The friendship graph F_5.

Therefore the friendship graph $F_2^{(m)}$ is difference cordial iff $m \leq 5$.

Chapter 3

Some Results and Examples on Difference Cordial Graphs

In this chapter we introduce some results on difference cordial graphs and describe the difference cordial labeling for some families of graphs.

3.1 Introduction

Through this chapter we will deal with finite simple undirected graphs. By $G = (V, E)$ we mean a finite undirected graph with p vertices and q edges where $p = |V|$ and $q = |E|$. For standard terminology and notations we follow Harary [14], and for more details of labeling see [12]

R. Ponraj, S. Sathish Narayanan and R. Kala[25], firstly, introduced the concept of difference cordial labeling in 2013. After that, they introduced many concepts and studied some types of graphs that have this kind of labeling such as: path, cycle, complete graph, complete bipartite graph, bistar, wheel, web, sunflower graph, lotus inside a circle, pyramid, permutation graph, book with n pentagonal pages, $t-$fold, wheel, double fan and some more standard graphs have been investigated in [23, 24, 25, 26, 27]. Within this area M. A.Seoud and Shakir M. Salman introduced some results and investigated some difference cordial graphs: ladder, step ladder,two sided step ladder,diagonal ladder,triangular ladder, grid graph and some types of one-point union graphs[34]

Definition 6. *[14] The line graph $L(G)$ of a graph G has a vertex for each edge of G, and two of these vertices are adjacent if and only if the corresponding edges in G have a common vertex.*

Definition 7. *[14] A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face; we usually choose this face to be exterior. An outerplanar graph G is maximal outerplanar if no line can be added without losing outerplanarity.*

The maximal outerplanar graph is denoted by MOG in this thesis. The MOG has the following propositionerties:

Lemma 3.1. *[14] Let G be an MOG with n vertices;$n \geq 3$, then:*

1. *there are $2n-3$ edges, in which there are $n-3$ chords;*
2. *there are $n-2$ inner faces and each inner face is triangular;*
3. *there are at least two vertices with degree 2;*
4. *connectivity of G, $k(G)$ is equal to 2.*

Definition 8. *[2] Let $G = (V, E)$ be a graph and $u \in V$ a vertex of G. The open neighborhood of u or just the neighborhood of u, denoted by $N_G(u)$ or just $N(u)$, is the set of all of the neighbors of u in G. Likewise, the closed neighborhood of u, denoted by $N_G[u]$ or just $N[u]$, is the set of neighbors of u together with u itself.*

Definition 9. *[43] Duplication of a vertex v_i by a new edge $e = v_i^{'}v_i^{''}$ in a graph G produces a new graph $G^{'}$ such that $N(v_i^{'}) \cap N(v_i^{''}) = \{v_i\}$.*

Definition 10. *[9] A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex, shell graphs are denoted as $C(n, n-3)$.*

Definition 11. *[42] A bow graph is defined to be a double shell in which each shell has any order.*

Definition 12. *[42] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.*

Definition 13. *[12] Define the shell-flower graph as k copies of the union of the shell $C(n, n-3)$ and K_2 where one end vertex of K_2 is joined to the apex of the shell. Figure 3.7 shows this type of graphs.*

3.2 Some Results

Proposition 3.2. *The graph $G = (V, E)$ is difference cordial if and only if there exist some disjoint paths, such that their total length is more than or equal to $\left\lfloor \frac{1}{2} |E| \right\rfloor$.*

Proof. Let $G = (V, E)$ be a difference cordial graph with a mapping f, then $|e_f(1) - e_f(0)| \leq 1$, i.e., $e_f(1) - 1 \leq e_f(0) \leq e_f(1) + 1$ and $e_f(1) + e_f(0) = |E|$.

If $|E|$ is even, then $e_f(1) = e_f(0)$ *and* $e_f(1) = \frac{1}{2}|E|$, this means $\frac{1}{2}|E|$ edges join the vertices labeled $i, i+1$.Then there are some paths such that, the sum of their lengths is $\frac{1}{2}|E|$ which is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

If $|E|$ is odd, then $e_f(1) = e_f(0) + 1$ or $e_f(1) = e_f(0) - 1$, if $e_f(1) = e_f(0) + 1$, then $e_f(1) = \left\lceil \frac{1}{2}|E| \right\rceil$, thus there are $\left\lceil \frac{1}{2}|E| \right\rceil$ edges join the vertices labeled $i, i+1$; in other words, there are some paths such that, the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

If $e_f(1) = e_f(0) - 1$, then $e_f(1) = \left\lfloor \frac{1}{2}|E| \right\rfloor$, thus there are $\left\lfloor \frac{1}{2}|E| \right\rfloor$ edges join the vertices labeled $i, i+1$, this means the existence of some disjoint paths, as above, the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

Thus, if $G = (V, E)$ is a difference cordial graph, then there exist some disjoint paths, the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$. It is necessary that the paths are disjoint otherwise there are three paths having a common vertex, which is impossible, since if this common vertex has the label x, two of the adjacent vertices should have the labels $x - 1$, $x + 1$. But the third vertex can't take either $x - 1$ or $x + 1$, but something else.

Suppose there exist some disjoint paths, the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$ on the graph $G = (V, E)$. If there is only one such path and we labeled its vertices by $i, i + 1, ..., \left\lfloor \frac{1}{2}|E| \right\rfloor$, then all edges of this path are labeled 1.

If there are two disjoint paths their lengths are k and h where $k+h$ is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$, then we label these paths by $j, j+1, ..., j+k$ and $t, t+1, ..., t+h$, respectively, hence there are $k+h$ edges labeled 1, continue this procedure with all paths.

Then $G=(V,E)$ is difference cordial. □

Proposition 3.3. *If the graph $G=(V,E)$ is semi-Hamiltonian, then G is a difference cordial graph if and only if the length of the semi-Hamiltonian path is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.*

Proof. Let G be a (p,q) graph containing a semi-Hamiltonian path where $q=|E|$, and let G be a difference cordial graph, since G contains a semi-Hamiltonian path, we can label it such that $e_f(1)=p-1$ and according to Proposition 2.1;

$$q \leq 2p-1 \Longrightarrow p-1 > \left\lfloor \frac{1}{2}q \right\rfloor - 1$$

G is difference cordial implies the length of the semi-Hamiltonian path is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

Suppose G contains a semi-Hamiltonian path whose length is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$, then G is difference cordial by Proposition 3.2. □

Proposition 3.4. *Every connected graph $G(p,q)$ with $q=2p-1$ is difference cordial if and only if G is semi-Hamiltonian.*

Proof. Let $G(p,q)$ be undirected simple connected with $q = 2p-1$.

Suppose G is a difference cordial graph then there exists labeling f such that $|e_f(1) - e_f(0)| \leq 1$. From Proposition 3.2 there is a path with length at least $p-1$, but by [42] the maximum length is $p-1$, then this path must pass through all vertices of graph G, i.e., G is semi-Hamiltonian.

If G is a semi-Hamiltonian graph, we label the vertices on a semi-Hamiltonian path by a sequence of integers $1, 2, ..., p$ we will get $p-1$ edges labeled 1 and the other edges are labeled 0, this means G is difference cordial. □

Corollary 3.5. *The Peterson graph is difference cordial.*

Proof. Direct consequence of Proposition 3.3. □

Proposition 3.6. *Every biconnected outerplanar graph is a difference cordial graph.*

Proof. Let G (p,q) be a biconnected outerplanar graph, then G is a Hamiltonian graph thus there is a semi-Hamiltonian path in G.

Case 1: If G is a maximal outerplanar graph, then $q = 2p-3$ thus $q \leq 2p-1$ and we label the vertices in this path by a sequence of integer numbers so we get $p-1$ of edges labeled 1, and other edges will be labeled 0, thus $e_f(0) = p-2$, means G is difference cordial.

Case 2: If G is not a maximal outerplanar graph, then $q \leq 2p-3$; thus $q \leq 2p-1$, then the semi-Hamiltonian path is of length $p-1$, which is more than $\left\lfloor \frac{1}{2}q \right\rfloor$, then by Proposition 3.3, G is difference cordial.

Then from **Case 1** and **Case 2** every biconnected outerplanar graph is difference cordial. □

Example 3.7. *The following outerplanar graphs with their difference cordial labeling are shown in Figure 3.1 and Figure 3.2 respectively,*
$p = 12$, $q = 21$ *and* $e_f(0) = 10, e_f(1) = 11$

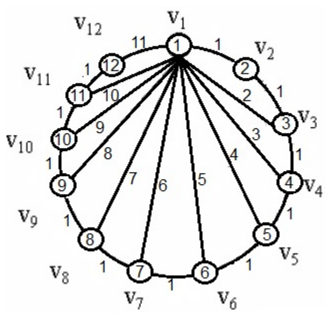

Figure 3.1: A difference cordial labeling for the maximal outerplanar graph with 12 vertices.

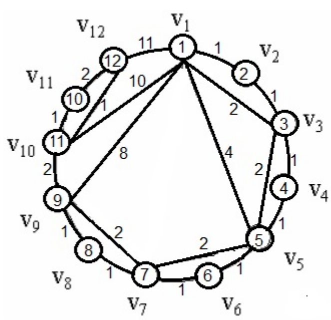

Figure 3.2: An outerplanar graph with 12 vertices and 20 edges.

$p = 12$, $q = 20$ *and* $e_f(0) = 10, e_f(1) = 10.$

Proposition 3.8. *1. The line graph $L(G)$ for any graph G with $\delta(G) \geq 3$ cannot be difference cordial.*

2. The line graph $L(G)$ for any graph G with $d(G) \geq 3$ cannot be difference cordial.

Proof. 1. Let $G(p,q)$ be a simple undirected graph with $\delta(G) \geq 3$, and $L(G)$ its line graph.
The number of vertices on $L(G)$ is equal to q, each of these edges in $L(G)$ has on its ends two vertices with degree more than or equal to 3, then the vertex corresponding with this edge in the line graph $L(G)$ will be of degree more than 4, then $L(G)$ is a graph with $\delta(G) \geq 4$ then by Proposition 2.3, it cannot be difference cordial.

2. It follows directly from 1.

□

Remark 3.9. *The union of two disjoint difference cordial graphs need not be difference cordial.*

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two disjoint difference cordial graphs, where f_1 and f_2 are their labelings, then $|e_{f_1}(0) - e_{f_1}(1)| \leq 1$ and $|e_{f_2}(0) - e_{f_2}(1)| \leq 1$.

Since if $e_{f_1}(0) = e_{f_1}(1) + 1$, and $e_{f_2}(0) = e_{f_2}(1) + 1$, and G_1, G_1 are disjoint graphs, then $G_1 \cup G_2$ has $q = q_1 + q_2$ and $p = p_1 + p_2$, then $e_{f_1}(0) + e_{f_2}(0) = e_{f_1}(1) + e_{f_2}(1) + 2$, hence $G_1 \cup G_2$ is not difference cordial. □

Example 3.10. *The following two disjoint difference cordial graphs are shown in Figure 3.3.*

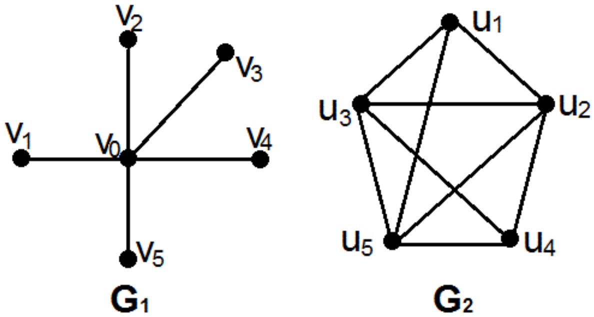

Figure 3.3: Two disjoint difference cordial graphs.

3.3 Difference Cordial Labeling for Some Families of Graphs

In this section we introduce difference cordial labeling for some types of graphs.

3.3.1 Graph Obtained by Duplication of Vertex by an Edge

Here we discuss only the graph obtained by duplication of each vertex of C_n by an edge.

Proposition 3.11. *The graph obtained by duplication of each vertex of C_n by an edge is difference cordial.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n and G be the graph obtained by duplication of each vertex v_i of the cycle C_n by an edge u_iu_{i+1} $(1 \leq i \leq n)$.

Then $V(G) = V(C_n) \cup \{u_1, u_2, ..., u_{2n}\}$ and
$E(G) = E(C_n) \cup \{u_{2i-1}v_i, u_{2i}v_i, u_{2i}u_{2i-1}; 1 \leq i \leq n\}$, then $V(G) = 3n$ and $E(G) = 4n$.

Define the mapping $f : V(G) \longrightarrow \{1, 2, ..., 3n\}$ by:

$$f(u_i) = i \quad \& \quad f(v_i) = \begin{Bmatrix} i+1+2n & , \ i \neq n \\ 2n+1 & , \ i = n \end{Bmatrix}$$

From the definition of $f(u_i)$ there are n edges labeled 1 and from $f(v_i)$ there are $n-1$ edges labeled 1, also the edge $u_{2n}v_n$ is labeled 1, then $e_f(1) = 2n$ and G is a difference cordial graph. $\square$

Example 3.12. *The graph obtained by duplication of vertex of C_7 by an edge with its difference cordial labeling is shown in Figure 3.4:*

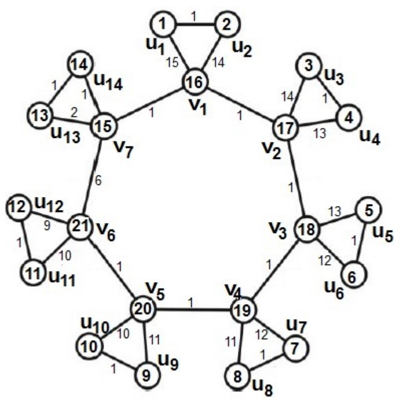

Figure 3.4: The difference cordial labeling for the graph obtained by duplication of vertex of C_7 by an edge.

3.3.2 Bow Graphs

The bow graph $G(p, q)$ could be described as follows: In graph G, the shell that is present to the left of the apex is called the left wing and the shell that is present to the right of the apex and it is considered as the right wing.

Figure 3.5 shows the bow graph with shells of orders m and n excluding the apex.

Proposition 3.13. *All bow graphs are difference cordial.*

Proof. Let G be a bow graph with two shells of orders m and n excluding the apex. Then, the number of vertices in G is $p = m + n + 1$ and the number of edges $q = 2(m+n-1)$. The apex of the bow graph is denoted by v_0, denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$.

Define the mapping of labeling

$f : V \longrightarrow \{1, 2, ..., m + n + 1\}$ by:

$$f(v_i) = \left\{ \begin{array}{cc} i & , \ i \neq 0 \\ m+n+1 & , \ i = 0 \end{array} \right\}$$

From the above definition we see there are $m-1$ edges labeled 1 and m edges labeled 0 in the right wing of the bow graph, also there are n edges labeled 1 and $n-1$ edges labeled 0 in the left wing of the bow graph. Then $|e_f(1)| = |e_f(0)| = m+n-1$, which implies the bow graph is difference cordial. □

Example 3.14. *The bow graph G with two wings having m, n vertices respectively, with its difference cordial labeling is shown in Figure 3.5.*

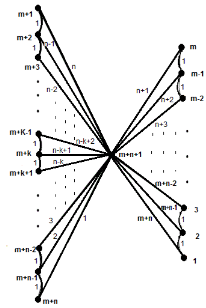

Figure 3.5: The bow graph with $m + n + 1$ vertices and its difference cordial labeling.

3.3.3 Butterfly Graphs

Proposition 3.15. *The butterfly graphs are difference cordial.*

Proof. Let G be a butterfly graph with shells of orders m and n excluding the apex, then the number of vertices in G is $p = m+n+3$ and the number of edges $q = 2(m + n)$. The apex of the butterfly graph is denoted as v_0, denote the vertices in the right wing of the butterfly graph from bottom to top by $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the pendant vertices in the pendant edges are denoted by v_{m+n+1}, v_{m+n+2}.

Define the mapping of labeling $f : V \longrightarrow \{1, 2, ..., m + n + 3\}$ by:

$$f(v_i) = \left\{ \begin{array}{cl} i & , 1 \leq i \leq n + m \\ m + n + 1 & , i = 0 \\ i + 1 & , n + m < i \leq n + m + 2 \end{array} \right\}$$

From the above definition we see there are $m-1$ edges labeled 1 and m edges labeled 0 in the right wing of the bow graph, also there are n edges labeled 1 and $n-1$ edges labeled 0 in the left wing of the butterfly graph, while the pendant edges are labeled 1 and 0.

Then $|e_f(1)| = |e_f(0)| = m+n$, which implies the butterfly graph is difference cordial. □

Example 3.16. *The butterfly graph G with two wings having m, n vertices respectively, and its difference cordial labeling is shown in Figure 3.6.*

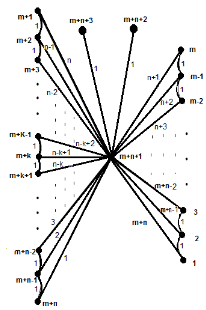

Figure 3.6: A difference cordial labeling for the butterfly graphs.

3.3.4 Shell-Flower Graphs

From the definition of the shell-flower graph it contains k copies of the shell $C(n, n-3)$ and k copies of K_2 where one vertex of K_2 is joined to the apex of the shell and each shell in the shell-flower graph is called a petal, hence it consists of k petals and k pendant edges.

Proposition 3.17. *The shell-flower graph cannot be difference cordial when $k \geq 3$ for all n.*

Proof. Let $G(p,q)$ be the shell-flower graph with k petals in each one $C(n,n-3)$, and then $p = nk+1$, and $q = 2k(n-1)$ as in Figure 3.7.

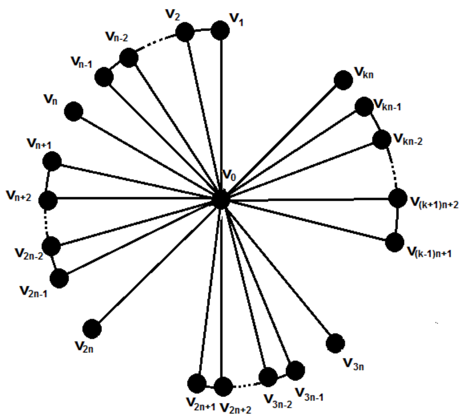

Figure 3.7: A shell-flower graph with k petals.

The apex vertex has degree $p-1$, then for any labeling of vertices there are at least $p-3 = nk-2$ edges labeled 0 but the number of the total other edges is:

$$2k(n-1) - nk + 2$$
$$= (nk-2) + (4-2k),$$

assume all these edges are labeled as 1. Then for all $k \geq 3$, $e_f(0) \geq e_f(1) + 2$. Hence the shell-flower graph is not difference cordial for all $k \geq 3$. □

Example 3.18. *The shell-flower graph with two petals $C(9,6)$ is shown in Figure 3.8.*

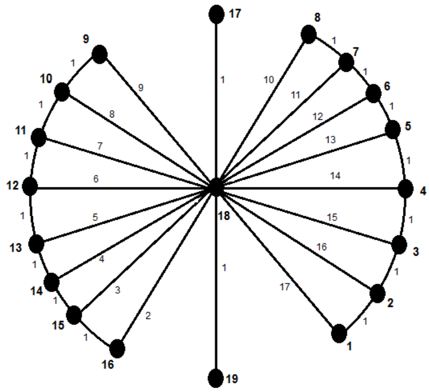

Figure 3.8: A difference cordial labeling for the shell-flower graph with two petals.

3.3.5 One-Point Union of Complete Graphs

In this subsection we discuss the difference cordial labeling of the one-point union of m complete graphs K_n of order n, as in Figure 3.9.

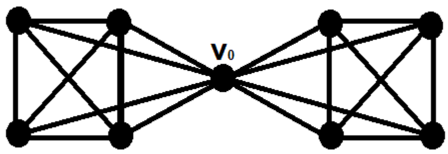

Figure 3.9: The graph $K_5^{(2)}$.

Proposition 3.19. *Let $K_n^{(m)}$ be the one-point union of m complete graphs K_n.*

1. *$K_2^{(m)}$ is difference cordial when $m \leq 5$.*

2. *$K_3^{(m)}$ is difference cordial when $m \leq 5$.*

3. *$K_4^{(m)}$ is difference cordial when $m \leq 2$.*

4. *$K_n^{(m)}$ is not difference cordial for all $n \geq 5$.*

Proof. 1. $K_2^{(m)}$ is star graph from [37] it is difference cordial when $m \leq 5$.

2. $K_3^{(m)}$ is friendship graph from [37] it is difference cordial when $m \leq 5$.

3. Let $G = K_4^{(m)}$, then G has $3m+1$ vertices and $6m$ edges, so there is one vertex say v_0 adjacent with all other vertices in G, so the graph $G - v_0$ consists of m components each component is a triangle which consists of at most 2 edges labeled 1 and at least one edge labeled 0, i.e., there are at most $2m$ edges labeled 1 and at least m edges labeled 0. But v_0 is adjacent with all other vertices, hence its degree is $3m$, then G contains at least $4m - 2$ edges labeled 0 and at most $2m + 2$ edges labeled 1 and we have:

$$\begin{aligned} e_f(0) - e_f(1) &\geq (4m-2) - (2m+2) \\ &\geq 2m - 4 \end{aligned}$$

If $m \geq 3$ then $e_f(0) - e_f(1) \geq 2$,thus $K_4^{(m)}$ is not difference cordial for all $m \geq 3$.

4. In graph $K_n^{(m)}$,$\delta(K_n^{(m)}) \geq 4$ when $n \geq 5$, then by Proposition 2.3, $K_n^{(m)}$ is not difference cordial.

□

Chapter 4

Some Results on Divisor Cordial Graphs

In this chapter we introduce some results on divisor cordial graphs and describe the divisor cordial labeling for some families of graphs.

4.1 Introduction

In this chapter by a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [14] . Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [12]. The concept of cordial labeling was introduced by Cahit [8], in [46], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graphs are presented in [29, 18, 22, 19, 20, 45, 44, 46, 47]. The brief summaries of Definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [7].

Definition 14. *[46] Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.*

Definition 15. *[14] The neighborhood of a vertex u is the set $N_u(G)$ consisting of all vertices v which are adjacent with u. The closed neighborhood is $N_u[G] = N_u(G) \bigcup \{u\}$.*

Definition 16. *[17] The Jelly fish graph $J(m, n)$ is obtained from a $4-$cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.*

Definition 17. *[9] A shell graph is defined as a cycle C_n with $(n - 3)$ chords sharing a common end point called the apex, shell graphs are denoted as $C(n, n - 3)$.*

Definition 18. *[42] A bow graph is defined to be a double shell in which each shell has any order.*

Definition 19. *[9] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.*

4.2 The Results

Proposition 4.1. *For any simple graph $G(p,q)$, the maximum value of $e_f(1)$ is $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$, where $p \geq 4$.*

Proof. Let $G(p,q)$ be a simple connected graph and let the vertex v_k be of maximum degree $\triangle(G)$, if we labeled this vertex by 1 then we will achieve $\triangle(G)$ edges labeled 1, and from division algorithm the maximum number of the multiples of labels of vertices are:

for 2 is $\lfloor \frac{p}{2} \rfloor - 1$,

for 3 is $\lfloor \frac{p}{3} \rfloor - 1$,

for 4 is $\lfloor \frac{p}{4} \rfloor - 1$,

.

.

.

for $\lfloor \frac{p}{2} \rfloor$ is $\lfloor \frac{p}{\lfloor \frac{p}{2} \rfloor} \rfloor - 1$ which must equal 1

hence the maximum value for $e_f(1)$ equals $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$ in any graph $G(p,q)$. $\square$

Corollary 4.2. *For each $r-$regular graph the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$; where $k = \lfloor \frac{p}{r+1} \rfloor$ and $p \geq 4$.*

Proof. Let $G(p,q)$ be an $r-regular$ graph then $\triangle(G) = r$, and for each vertex v in graph G the maximum number of edges that label 1 in $N_v(G)$ is r, hence for all i in which $\lfloor \frac{p}{i} \rfloor - 1 \geq r$ we reduced it to r.

But from Proposition 4.1 the maximum value of $e_f(1)$ is

$\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$, then the maximum value in an $r-regular$ graph is:

$$= r + \sum_{i=2}^{k}(r) + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

$$= r + (k-1)r + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

$$= kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

□

Proposition 4.3. *For any divisor Cordial graph $G(p,q)$, $q \leq 2(\triangle(G) + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor} \lfloor \frac{p}{i} \rfloor) + 3$, where $p \geq 6$.*

Proof. Let $G(p,q)$ be a divisor cordial graph, then $|e_f(0) - e_f(1)| \leq 1$, means $e_f(0) = e_f(1) - 1$ or $e_f(0) = e_f(1)$ or $e_f(0) = e_f(1) + 1$,

by proposition 4.1,

$$q \leq 2e_f(1) + 1$$

$$q \leq 2(\triangle\ G + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor - 1)) + 1$$

$$q \leq 2(\triangle\ G + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor)) + 3$$

□

4.3 Divisor Cordial Labeling for Some Families of Graphs

In this section we introduce the divisor cordial labeling for some types of graphs.

4.3.1 The Jelly Fish Graph

Proposition 4.4. *For $m, n \geq 1$, Jelly fish graph $J(m, n)$ is a divisor cordial graph.*

Proof. Let $G(V, E) = J(m, n)$. Then G has $(m + n + 4)$ vertices and $(m + n + 5)$ edges.
Without losing of generality, Let $m \leq n$. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}, V_2 = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = E_1 \cup E_2$, where $E_1 = \{xu, uy, yv, vx, xy\}, E_2 = \{uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define $f : V \to \{1, 2, ..., (m + n + 4)\}$ as follows:

$f(u) = 1, f(v) = 2, f(x) = m + n + 4, f(y) = m + n + 3$ and

$f(u_i) = 2(i + 1); i = 1, 2, ..., m$,

$$f(v_i) = \begin{cases} 2i + 1 & , i = 1, 2, ..., m \\ i + m + 2 & , i = m + 1, m + 2, ..., n \end{cases}$$

From the function f there are $m + 2$ edges labeled 1 sice $f(u) = 1$, and since $f(V) = 2$ there are exactly $\lfloor \frac{1}{2}(n - m) \rfloor$ of pendent edges from v labeled 1 and only one from vx or vy. means $e_f(1) = m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor$ and

$$e_f(0) = m + n + 5 - (m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor)$$
$$= n + 2 - \lfloor \frac{1}{2}(n - m) \rfloor,$$

Case 1: m, n are odd

The $|E|$ and $\lfloor n - m \rfloor$ are even, hence, $|e_f(0) - e_f(1)| = 1$

Case 2: m, n are even

The $|E|$ is odd and $\lfloor n - m \rfloor$ is even, hence, $|e_f(0) - e_f(1)| = 1$

Case 3: m is odd and n is even

The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Case 4: m is odd and n is even

The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Then from Case 1, Case 2, Case 3 and Case 4 the jelly fish graph is divisor cordial. □

Example 4.5. *The jelly fish graph* $j(6, 11)$ *and its divisor cordial labeling are shown in Figure4.1*

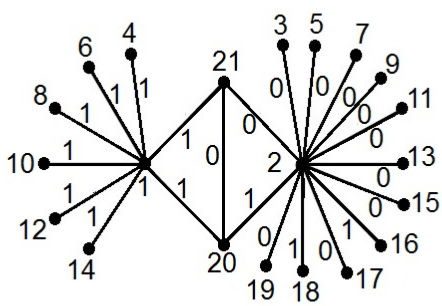

Figure 4.1: A Jelly fish graph $j(6, 11)$ and its divisor cordial labeling

4.3.2 The shell and The Bow Graph

Proposition 4.6. *Every shell graph is divisor cordial.*

Proof. Let $G = (V, E)$ be a $C(n, n-3)$ graph with $|V| = n$, then $|E| = 2n - 3$ means $|E|$ is an odd number, and let v_0 be the apex and $v_1, v_2, ..., v_{n-1}$ other its vertices.

Define the labeling $f : V \longrightarrow \{1, 2, ..., n\}$ as:

$f(v_0) = 2,\ f(v_1) = 1$ and other vertices by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, & \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, & \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \end{array}$$

In this labeling, there are $\lceil \frac{n-1}{2} \rceil$ edges label 1 passing through v_0, but other edges not passing through the apex make a path, hence there are also $\lfloor \frac{n-2}{2} \rfloor$ edges are labeled 1. Hence, $e_f(1) = \lceil \frac{n-1}{2} \rceil + \lfloor \frac{n-2}{2} \rfloor$

Case 1: n is odd, then $e_f(1) = \frac{n-1}{2} + \lfloor \frac{n-2}{2} \rfloor$ and $e_f(0) = \frac{n-1}{2} + \lceil \frac{n-2}{2} \rceil$

Case 2: n is even, then $e_f(1) = \lceil \frac{n-1}{2} \rceil + \frac{n-2}{2}$ and $e_f(0) = \lfloor \frac{n-1}{2} \rfloor + \frac{n-2}{2}$

In the two cases **Case 1** and **Case 2**, the difference between $e_f(1)$ and $e_f(0)$ is 1 which means the shell graph is divisor cordial. $\square$

Notice another divisor labeling for shell graph can found with fan graphs [46]

Example 4.7. *The shell graph* $C(13, 10)$ *and its divisor cordial labeling are shown in Figure 4.2*

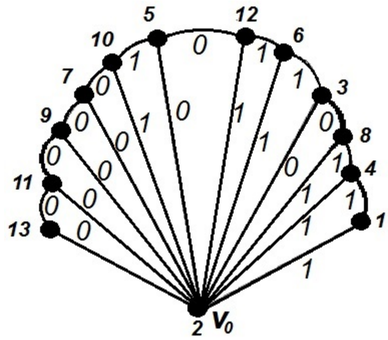

Figure 4.2: A shell graph $C(13, 10)$ and its divisor cordial labeling

Proposition 4.8. *All bow graphs are divisor cordial.*

Proof. Let G be a bow graph with two shells of orders m and n excluding the apex. Then the number of vertices in G is $p = m + n + 1$ and the edges $q = 2(m + n - 1)$. The apex of the bow graph is denoted by v_0, denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$. Without lose of generality suppose $m \leq n$.

Define the labeling $f : V \longrightarrow \{1, 2, ..., m + n + 1\}$ by:
$f(v_0) = 2,\ f(v_1) = 1$ and label the vertices of the wings by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $(2m-1)\cdot 2^{k_m} \leq n$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.

Let G' be a graph obtained from the bow graph G by adding the edge $v_m v_{m+1}$.

The graph G' has an odd number of edges and it is a shell graph, then by Proposition 4.6 the graph G' is divisor cordial. The graph $G = G' - v_m v_{m+1}$ with even edges, then G is divisor cordial since:

Case 1: If $m+n$ is even, then $e_f(0) = e_f(1) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 0.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then the deleted edge $v_m v_{m+1}$ is labeled 0.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-l}$ in the wings, by l where l is the smallest integer satisfying $f(v_{m+1}) = (2t-1)\cdot 2^{k_i}$ for some i, and shift the labels of the vertices $v_{m+n-l+1}, v_{m+n-l+2}, ..., v_{m+n}$, by $l+1$ and take it modulo $(m+n)$.

Case 2: If $m+n$ is odd, then $e_f(1) = e_f(0)+1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 1.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-1}$ in the wings, by one step and shift the label of vertex v_{m+n} by two and take it modulo $(m+n)$.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then the edge $v_m v_{m+1}$ is labeled 1.

Then the bow graph G with two wings of m and n vertices is a divisor cordial graph for each m and n. $\square$

Example 4.9. *The bow graph with two wings of* 13 *and* 16 *vertices respectively and its divisor cordial labeling are shown in Figure 4.3*

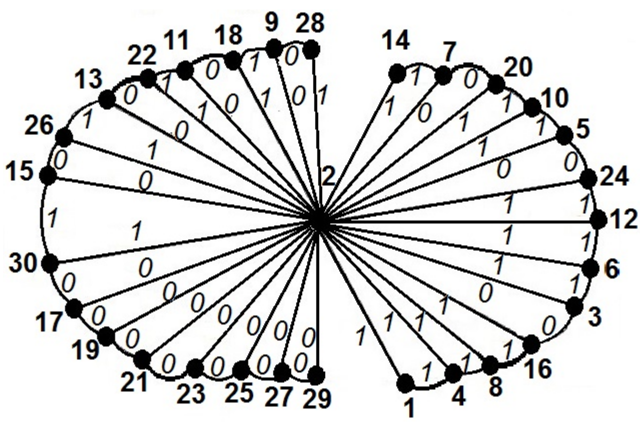

Figure 4.3: A bow graph with $m=13, n=16$ and its divisor cordial labeling

4.3.3 Butterfly Graphs

Proposition 4.10. *The butterfly graphs are divisor cordial.*

Proof. Let G be a butterfly graph with shells of orders m and n excluding the apex then the number of vertices in G is $p = m+n+3$ and the edges $q = 2(m+n)$. The apex of the butterfly graph is denoted as v_0,denote the vertices in the right wing of the butterfly graph from bottom to top as $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the vertices in the pendant edges are v_{m+n+1}, v_{m+n+2}.
Since the butterfly defined as a bow graph with exactly two pendent edges at the apex,then we define the labeling $f : V \longrightarrow \{1, 2, ..., m+n+3\}$ by: $f(v_0) = 2,\ f(v_1) = 1,\ f(v_{m+n+1}) = m+n+2,\ f(v_{m+n+2}) = m+n+3$ and labeled the vertices of the wings by the following:

$$\begin{array}{cccccc}
 & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\
3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\
5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

and we make the shift as in Proposition 4.8, for labeling of the vertices in the wings.
Since the only one of the numbers $m+n+2$ or $m+n+3$ must be even then the pendent edges will be labeled 1 and 0, hence the graph G is divisor cordial. $\square$

Example 4.11. *The butterfly graph G with two wings having $m = 9$, $n = 15$ vertices respectively, and its divisor cordial labeling is shown in Figure 4.4.*

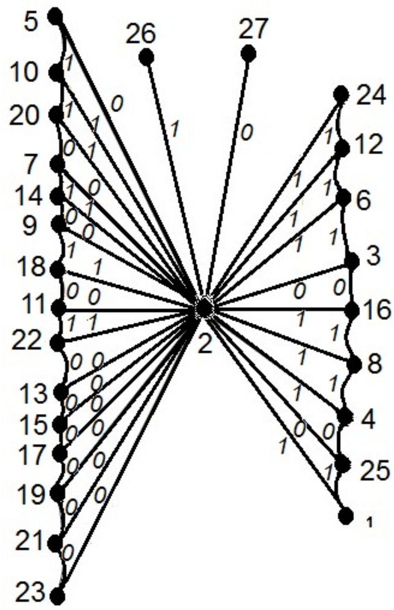

Figure 4.4: A divisor cordial labeling for the butterfly with 29 vertices

4.3.4 Friendship Graphs

The friendship graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. The friendship graph F_n is isomorphic to the windmill graph $Wd(3, n)$ [12]. The Friendship Theorem states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs [11]

Proposition 4.12. *The friendship graphs are divisor cordial.*

Proof. Let $G(p,q)$ be a friendship graph F_n then the number of vertices in G is $p = 2n+1$ and $q = 3n$ edges.
Define $f : V \to \{1, 2, ..., (2n+1)\}$ as follows:
$f(v_0) = 2,\ f(v_1) = 1,\ f(v_2) = 4,\ f(v_{2n}) = 2n+1$ and

$$f(v_i) = \begin{cases} i & , i = 3, 5, ..., 2n-1 \\ 2(i-1) & , i = 4, 6, 8, ..., 2(\lceil \frac{n}{2} \rceil + 1) \\ 4(\frac{i}{2} - \lfloor \frac{n}{2} \rfloor) & , i = 2(\lceil \frac{n}{2} \rceil + 1), 2(\lceil \frac{n}{2} \rceil + 1) + 2, ..., 2n-2 \end{cases}$$

from definition of f there are n edges labeled 1 incident from the central vertex, also, $\lceil \frac{n}{2} \rceil$ edges are labeled 1 on other edges of the friendship F_n which are in the first $\lceil \frac{n}{2} \rceil$ triangles and all others edges are labeled 0, then $e_1 = n + \lceil \frac{n}{2} \rceil$ and $e_0 = n + \lfloor \frac{n}{2} \rfloor$, hence, the friendship F_n is divisor cordial for all n. □

Example 4.13. *The friendship graph F_6 and its divisor cordial labeling are shown in Figure 4.5*

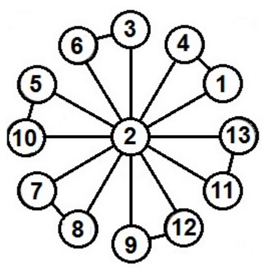

Figure 4.5: A friendship graph F_6 and its divisor cordial labeling.

Chapter 5

Divisor Cordial Labeling for Some Trees and Families of Graphs

In this chapter, we introduce divisor cordial labelings for some types of trees and some families of graphs.

5.1 Introduction

In this chapter by a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary[14]. Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [12]. The concept of cordial labeling was introduced by

Cahit [8], in [46], Varatharajan et al. introduce the concept of divisor cordial labeling of graph.

The divisor cordial labeling of various types of graphs are presented in [29, 18, 22, 19, 20, 45, 44, 46, 47]. The brief summaries of definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [7].

Definition 20. *[31]An $m-$star has a single root node with any number of paths of length m attached to it .*

Definition 21. *[3]A spider tree is a tree with at most one vertex of degree greater than 2. If such a vertex exists, it is called the branch point of the tree. A leg of a spider tree is any one of the paths from the branch points to a leaf of the tree.*

Definition 22. *[21] A $k-$distant tree consists of a main path called the spine, such that each vertex on the spine is joined by an edge to at most one path on $k-$vertices. Those paths are called tails. When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform $k-$distant tree.*

Definition 23. *[1] An olive tree has a root node with k branches attached: the i^{th} branch has length i.*

Definition 24. *[16] Let $G_1, G_2, ..., G_n$ be a family of disjoint stars. The tree obtained by joining a new vertex a to one pendant vertex of each star G_i is called a banana tree.*

5.2 The Results

Proposition 5.1. *All simple connected graphs with number of vertices less than eight are divisor cordial except* K_4.

Proof. In the following Figures: Figure 5.1, Figure 5.2 and Figure 5.3 We Shown all nonisomorphic graphs of $4, 5$ *and* 6 with their divisor cordial graphs except K_4 and in Appendix A we will show the graphs with 7 vertices.

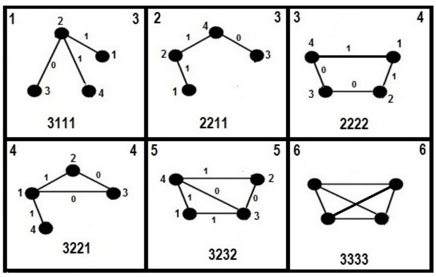

Figure 5.1: A divisor cordial labeling for all connected graphs with four vertices except K_4

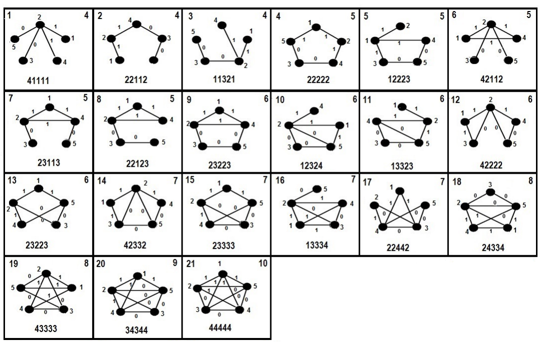

Figure 5.2: A divisor cordial labeling for all connected graphs with five vertices.

□

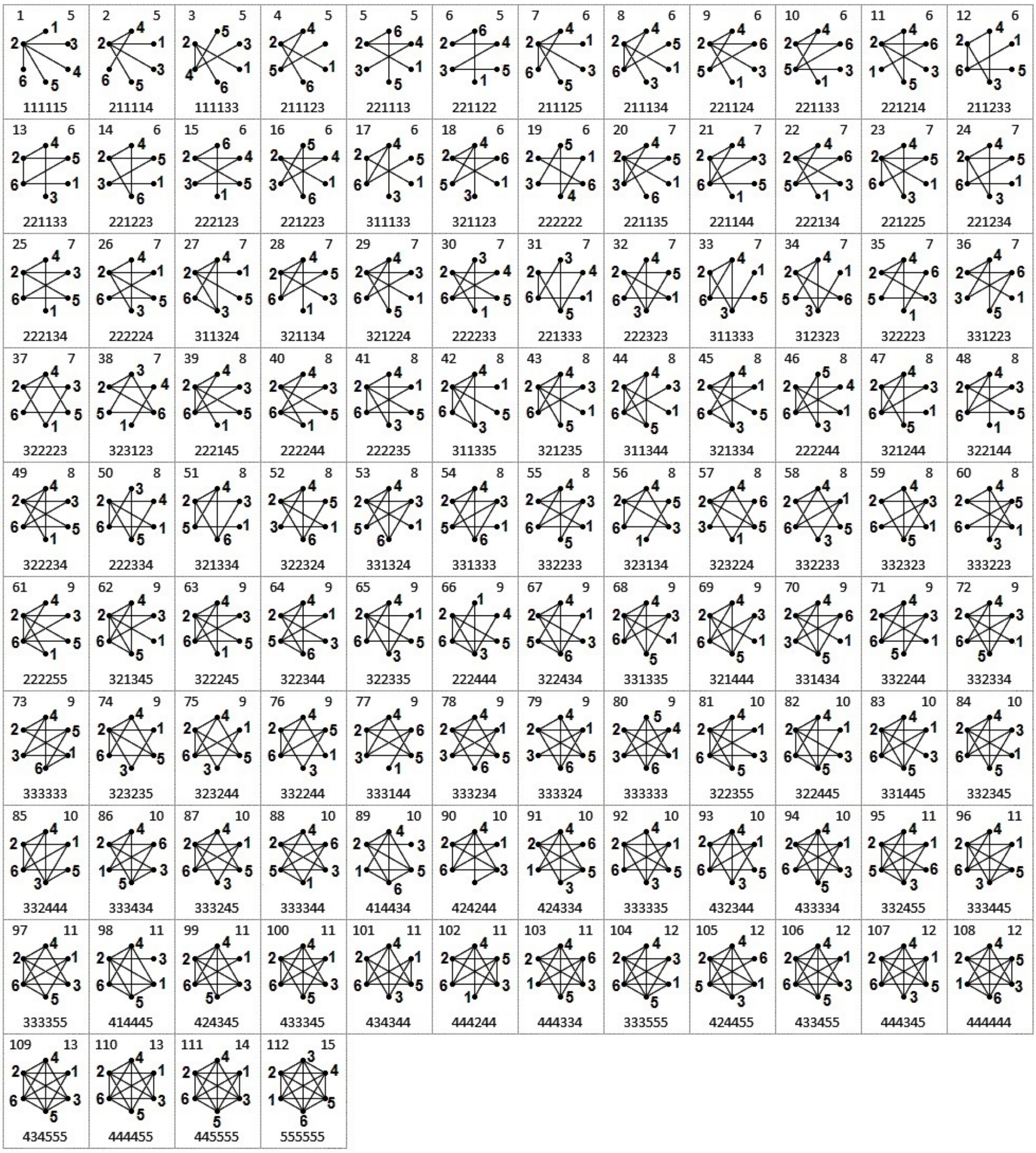

Figure 5.3: A divisor cordial labeling for all connected graphs with six vertices.

Proposition 5.2. *Every $r-$regular graph with $r > 6$ is not divisor cordial graph.*

Proof. Let $G(p,q)$ be an $r-$regular graph. The minimum number of edges in $r-$regular graph with $r > 6$ is 28 in the complete graph K_8, but from the Proposition 4.1 the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$; where $k = \lfloor \frac{p}{r+1} \rfloor$, that mean the maximum value of $e_f(1)$ in K_8 is:

$$
\begin{aligned}
& kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1), \quad k = \left\lfloor \frac{8}{8} \right\rfloor = 1 \\
& = 7 + \sum_{i=2}^{\lfloor \frac{8}{2} \rfloor} (\left\lfloor \frac{8}{i} \right\rfloor - 1) \\
& = 7 + (\left\lfloor \frac{8}{2} \right\rfloor - 1) + (\left\lfloor \frac{8}{3} \right\rfloor - 1) + (\left\lfloor \frac{8}{4} \right\rfloor - 1) \\
& = 12
\end{aligned}
$$

We notice that: $\frac{1}{2} \cdot (28) \nleq 12 + 1$, which means the graph cannot be divisor cordial. Thus all $r-$ regular graphs with r more than 6 are not divisor cordial.

□

5.3 Divisor Cordial Labeling for Some Trees

In this section we introduce the divisor cordial labeling for some types of trees such as olive, spider, $m - stars$, banana and caterpillar tree.

5.3.1 Olive Tree

Proposition 5.3. *All olive trees are divisor cordial graphs.*

Proof. Let ol_n be an olive tree of n branches, then the numbers of its edges and vertices are $|E| = \frac{n(n+1)}{2}$ and $|V| = |E| + 1$, respectivly, denotes the vertices as: $v_0, v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{3,3}, ..., v_{n,1}, v_{n,2}, ..., v_{n,n}$.

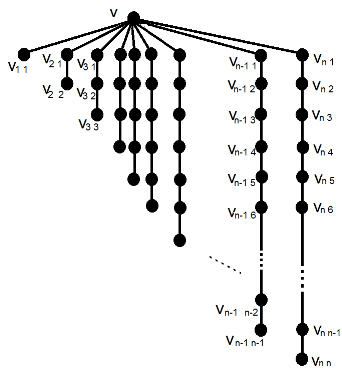

Figure 5.4: The olive tree ol_n

Define the labeling of vertices as: $f(v_0) = 2, f(v_{1,1}) = 1$ and label the vertices $v_{n,1}, v_{n,2}, ..., v_{n,n}, v_{n-1,1}, v_{n-1,2}, ..., v_{n-1,n-1}, v_{n-2,1}, ..., v_{2,1}, v_{2,2}$ by the numbers:

$$
\begin{array}{cccccc}
 & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, & \\
3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, & \\
5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots,
\end{array}
$$

Where $(2m-1)\cdot 2^{k_m} \leq n$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b; (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.

We will discuss the labeles of the vertices $v_{i,1}$ for the branches $n-1, n-2, ..., 2$ in two cases :

Case 1: If $f(v_{i,1})$ is odd then the edges $v_0v_{i,1}$ are labeled as 0

Case 2: If $f(v_{i,1})$ is even then the edges $v_0v_{i,1}$ are labeled as 1

That mean the labelng is same as chain and hold $|e_f(0) - e_f(1)| \leq 1$.

□

Example 5.4. *The divisor cordial labeng of olive tree ol_6 is shown in the Figure 5.5.*

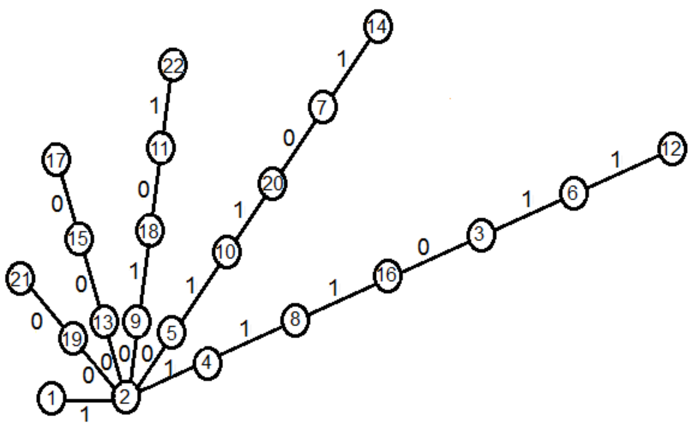

Figure 5.5: A divisor cordial labeling for olive tree ol_6

5.3.2 Spider Tree

A tree is called a spider if it has a centeral vertex c with degree $k > 1$ while each of the other vertices is either a leaf or has degree 2. Thus, a spider is an amalgamation of k paths with various lengths. If it has x_1 paths with length a_1, x_2 paths with length a_2, and so on, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, ..., a_m^{x_m})$ where $x_1 + x_2 + ... + x_m = k$, any one of the paths from c to a leaf of T is called leg of the spider T. (see Figure 5.6).

Proposition 5.5. *Spider Trees are divisor cordial graphs.*

Proof. Let $SP(a_1^{x_1}, a_2^{x_2}, ..., a_m^{x_m})$ be a spider tree. Define the labeling of vertices by: label the center vertex as 2 and the leaf of the latest leg as 1 and proceeding with other vertices in this leg and other legs from long to short in the same way as in olive tree (proof of Proposition 5.3).

□

Example 5.6. *The divisor cordial labeling for the spider tree* $SP(4^2, 3^3)$, *shown in Figure 5.6*

5.3.3 m-stars Tree

An $m - star$ tree has a single root node with any number of paths of length m such that each path attached to the root nod.

Proposition 5.7. *All* $m - star$ *trees are divisor cordial graphs.*

Proof. Same as proof of Proposition 5.5 □

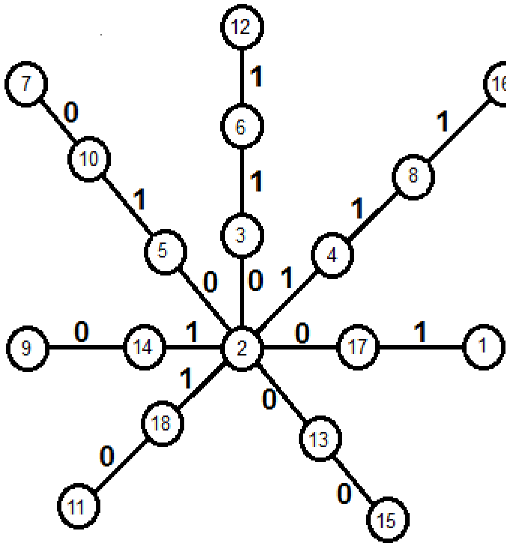

Figure 5.6: A divisor cordial labeling for spider tree

Example 5.8. *The divisor cordial labeling for* $4-$*star tree.*

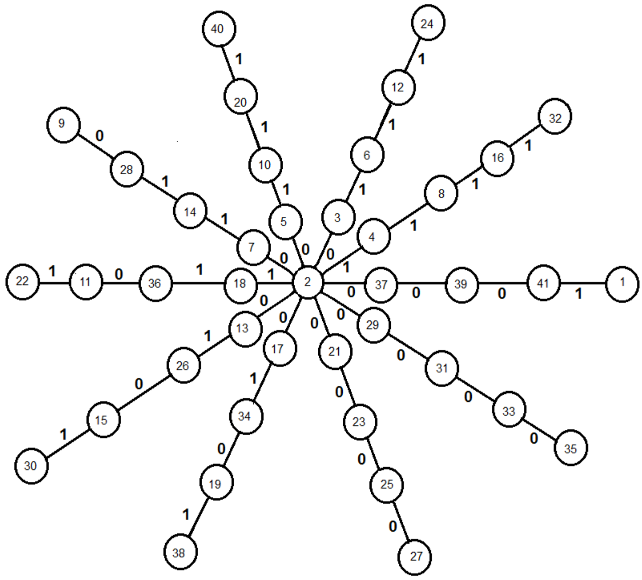

Figure 5.7: The $4-$star graph and its divisor cordial labeling

5.3.4 k-distant tree

A $k-$distant tree consists of a main path called the spine, such that each vertex on the spine is joined by an edge to at most one path on $k-$vertices. Those paths are called tails (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform $k-$distant tree. A uniform $k-$distant tree with odd number of vertices is called a uniform k-distant odd tree. A uniform $k-$distant tree with even number of vertices is called a uniform $k-$distant even tree. See Figure 5.8

Conjecture 1. *The uniform $k-$distant tree is divisor cordial.*

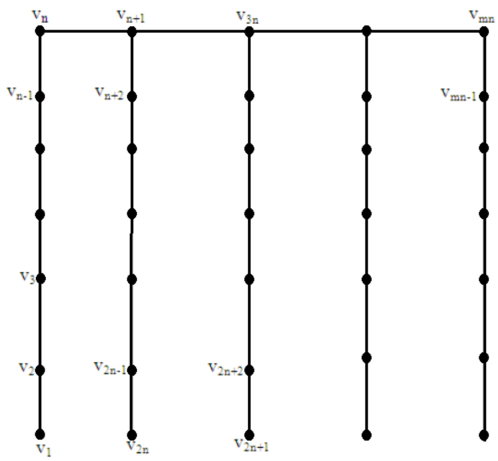

Figure 5.8: A divisor cordial labeling for the $k-$distant tree

5.3.5 Caterpillar Tree

Definition 25. *A caterpillar is a tree T such that for a maximum path P, all vertices are of distance at most one from P. Figure 5.9*

Conjecture 2. *All caterpillar trees are divisor cordial graphs.*

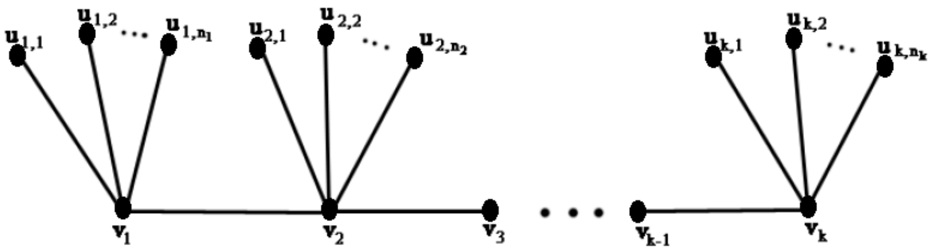

Figure 5.9: Caterpillar tree

5.3.6 Banana Tree

A banana tree is constructed by bringing multiple stars together at a single vertex by an edge from each one. Figure 5.10

Conjecture 3. *Banana trees are divisior cordial graphs.*

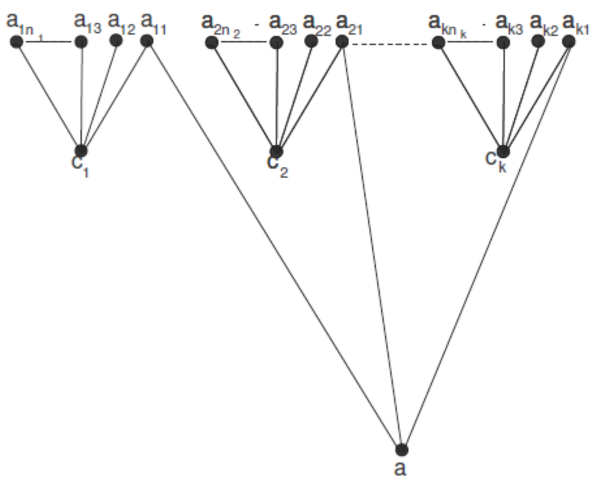

Figure 5.10: Banana tree

Bibliography

[1] V. J. Abhyankar and V. N. Bhat-Nayak, *Easiest graceful labeling of olive trees*, Bull. Bombay Math. Coll. **14** (2000), 16–25.

[2] G. Agnarsson and R. Greenlaw, *Graph theory: Modeling, applications, and algorithms*, pearson education inc ed.

[3] P. Bahl, S. Lake, and A. Wertheim, *Gracefulness of families of spiders*, Involve **3** (2010), 241–247.

[4] L.W. Beineke and R.J. Wilson, *Topics in algebraic graph theory, encyclopedia of mathematics and its applications, 102*, 7th edition ed.

[5] G. S. Bloom and S. W. Golomb, *Applications of numbered undirected graphs*, Proceeding of IEEE **65** (1977), no. 4, 562–570.

[6] Andreas Brandstdt, Van Bang Le, and Jeremy P. Spinrad, *Graph classes: A survey*, CSIAM Society for Industrial and Applied Mathematics, Philadelphia, 2004.

[7] D. M. Burton, *Elementary number theory*, 7th edition ed.

[8] I. Cahit, *Cordial graph :a weaker version of graceful and harmonious graphs*, Ars combinatoria **23** (1987), 201–207.

[9] P. Deb and N. M. Limaya, *On harmonious labelling of some cycle related graphs*, ARS COMBINATORIA **65** (2002), 177–197.

[10] Reinhard Diestel, *Graph theory*, Electronic addition, 2005.

[11] P. Erdös, A. Rényi, and V. T. Sós, *On a problem of graph theory*, Studia Sci. Math. Hungar **1** (1966), 215–235.

[12] J. A. Gallian, *Adynamic survey of graph labeling*, Electronic J comb **18** (2015), Ds6.

[13] R. L. Graham and N. J. A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Methods **1** (1980), 382–404.

[14] F. Harrary, *Graph Theory reading , massachusetts*, Addison-Wesley, 1969.

[15] D. A. Holton and J. Sheehan, *The petersen graph*, CAMBRIDGE UNIVERSITY PRESS, USA, 1993.

[16] M. Hussain, E. T. Baskoro, and Slamin, *On super edge-magic total labeling of banana trees*, Utilitas Math. **79** (2009), 243–251.

[17] K. Manimekalai and K. Thirusangu, *Pair sum labeling of some special graphs*, International Journal of Computer Applications **69** (2013), no. 8, 34–38.

[18] P. Maya and T. Nicholas, *Some new families of divisor cordial graph*, Annals of Pure and Applied Mathematics **5** (2014), no. 2.

[19] A. Nellai Murugan and V. Brinda Devi, *A study on path related divisor cordial graphs*, International Journal of Scientific Research **3**, no. 4.

[20] A. Nellai Murugan and M. Taj Nisha, *A study on divisor cordial labelling of star attached paths and cycles*, Indian Journal of Research **3** (2014), no. 3, 12–17.

[21] M. Murugan, *Magic properties of special class of trees*, International J.Math. Combin. **2** (2014), 81–88.

[22] A. Muthaiyan and P. Pugalenthi, *Some new divisorcordial graphs*, International Journal of Mathematics Trends and Technology **12** (2014), no. 2, 81–88.

[23] R. ponraj and S. S. Narayanan, *Difference coriality of some graphs obtained from double alternate snake graphs*, Global J. Math. Sciences: Theory and Practical **5** (2013), 167–175.

[24] R. Ponraj and S. S. Narayanan, *Further results on difference coedial labeling of corona graphs*, J Indian Acad Math **35** (2013), 217–235.

[25] R. Ponraj, S. S. Narayanan, and R. Kala, *Difference cordial labeling of graphs*, Global J. Math. Sciences: Theory and Practical **3** (2013), 193–201.

[26] R. Ponraj and S. Sathish Narayanan, *Difference cordial labeling of subdivision of snake graphs*, universal Journal of applied mathematics **2** (2014), no. 1, 40–45.

[27] R. Ponraj, S. Sathish Narayanan, and R. Kala, *A note on difference cordial graphs*, Palestine Journal of mathematics **4** (2015), no. 1, 189–197.

[28] N. Lakshmi Praanna, K. Sravanthi, and Nagalla Sudakar, *Applications of graph labeling in communication networks*, ORIENTAL JOURNAL OF COMPUTER SCIENCE TECHNOLOGY **7** (2014), no. 1, 139–145.

[29] P. Lawrence Rozario Raj and R. Valli, *Some new families of divisor cordial graphs*, International Journal of Mathematics Trends and Technology **7** (2014), no. 2, 94–102.

[30] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (International Symposium, Rome July 1966) Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[31] C. Sekar, *Studies in graph theory*, Ph.D. thesis, Madurai Kamaraj University, 2002.

[32] M. A. Seoud and M. E. Abdel-Aal, *On odd graceful graphs*, Ars Comb. **108** (2013), 161–185.

[33] M. A. Seoud and E. F. Helmi, *Some $\alpha-$graphs and odd graceful graphs*, Ars Combin. **101** (2011), 285–404.

[34] M. A. Seoud, A. E. I. Abdel Maqsoud, and J. Sheehan, *Harmonious graphs*, Util. Math. **47** (1995), 225–233.

[35] ______, *Gracefulness of the union of cycles and paths*, Ars Combin. **54** (2000), 283–292.

[36] M. A. Seoud and E. A. El Sahawi, *On variations of graceful labelings*, Ars Combin. **87** (2008), 127–138.

[37] M. A. Seoud and S. M. Salman, *On difference cordial graphs*, Mathematica Aeterna **5** (2015), 105–124.

[38] ______, *Some results and examples on difference cordial graphs*, Turk J Math **40** (2016), 417–427.

[39] M. A. Seoud and R. J. Wilson, *Some disgraceful graphs*, Int. J. Math. Ed. Sci. Tech. **24** (1993), 435–441.

[40] M. A. Seoud and M. Z. Youssef, *New families of graceful disconnected graphs*, Ars Combin. **57** (2000), 233–245.

[41] ______, *On harmonious graphs of order 6*, Ars Combin. **65** (2002), 155–176.

[42] E. H. Stanley and J. J. Jesintha, *Butterfly graphs with shell orders m and $2m+1$ are graceful*, Bonfring International Journal of Research in Communication Engineering **2** (2012), 1–5.

[43] S. K. Vaidya and L. Bijukmar, *Some new families of mean graphs*, Journal of Mathematics research **3** (2010), 169–176.

[44] S. K. Vaidya and N. H. Shah, *Further results on divisor cordial labeling*, Annals of Pure and Applied Mathematics **4** (2013), no. 2, 150–159.

[45] ______, *Some star and bistar related divisor cordial graphs*, Annals of Pure and Applied Mathematics **3** (2013), no. 1, 67–77.

[46] R. Varatharajan, S. Navanaeethakrishnan, and K. Nagarajan, *Divisor cordial graphs*, International J. Math. Combin. **4** (2011), 15–25.

[47] ______, *Special classes of divisor cordial graphs*, International Mathematical Forum **7** (2012), no. 35, 1737–1749.

[48] R.J. Wilson, *Introduction to graph theory*, 4th edition ed.

Appendix A

All Nonisomorphic Graphs with 7 Vertices and its Divisor Cordial Graphs

Connected graphs with 7 nodes

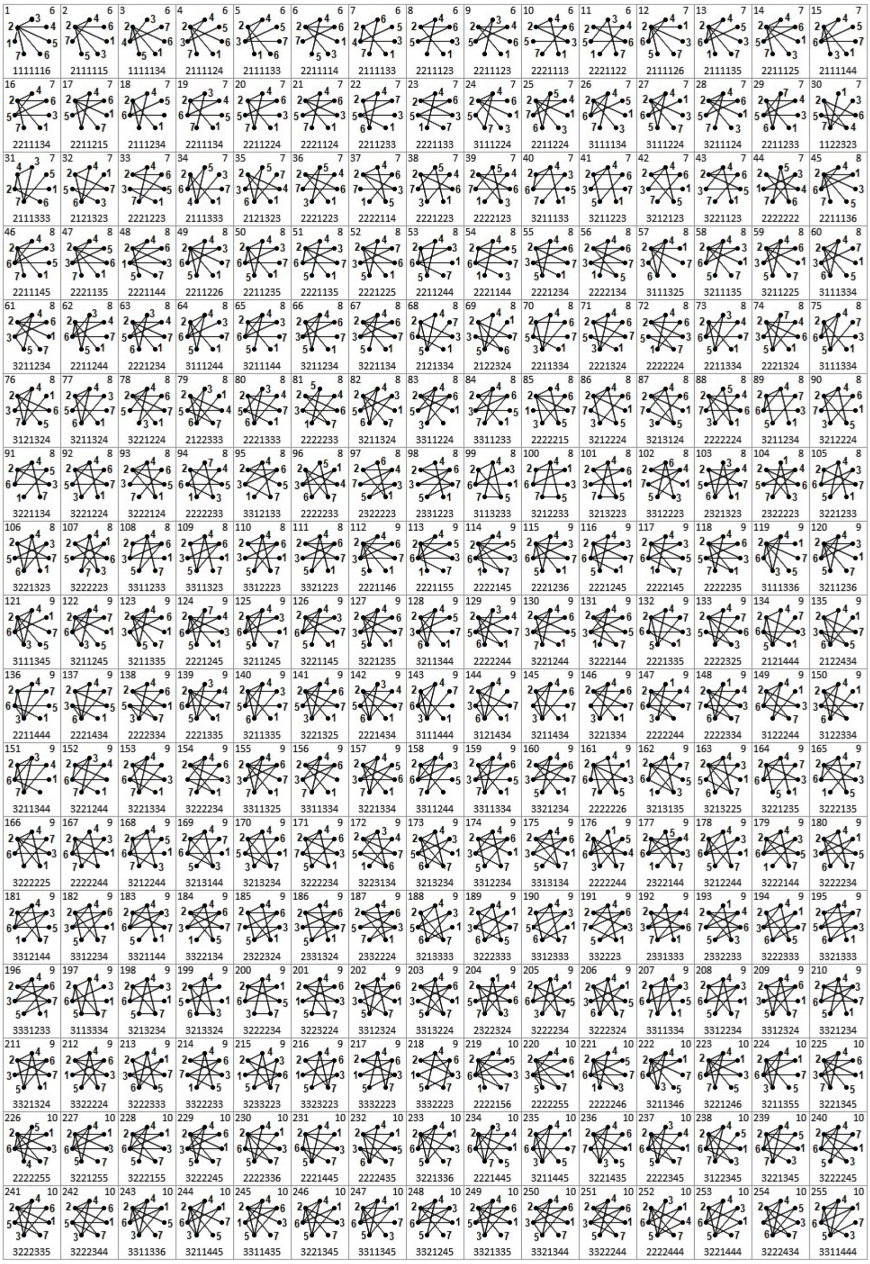

Figure A.1: A divisor cordial labelin g for all connected graphs with seven vertices

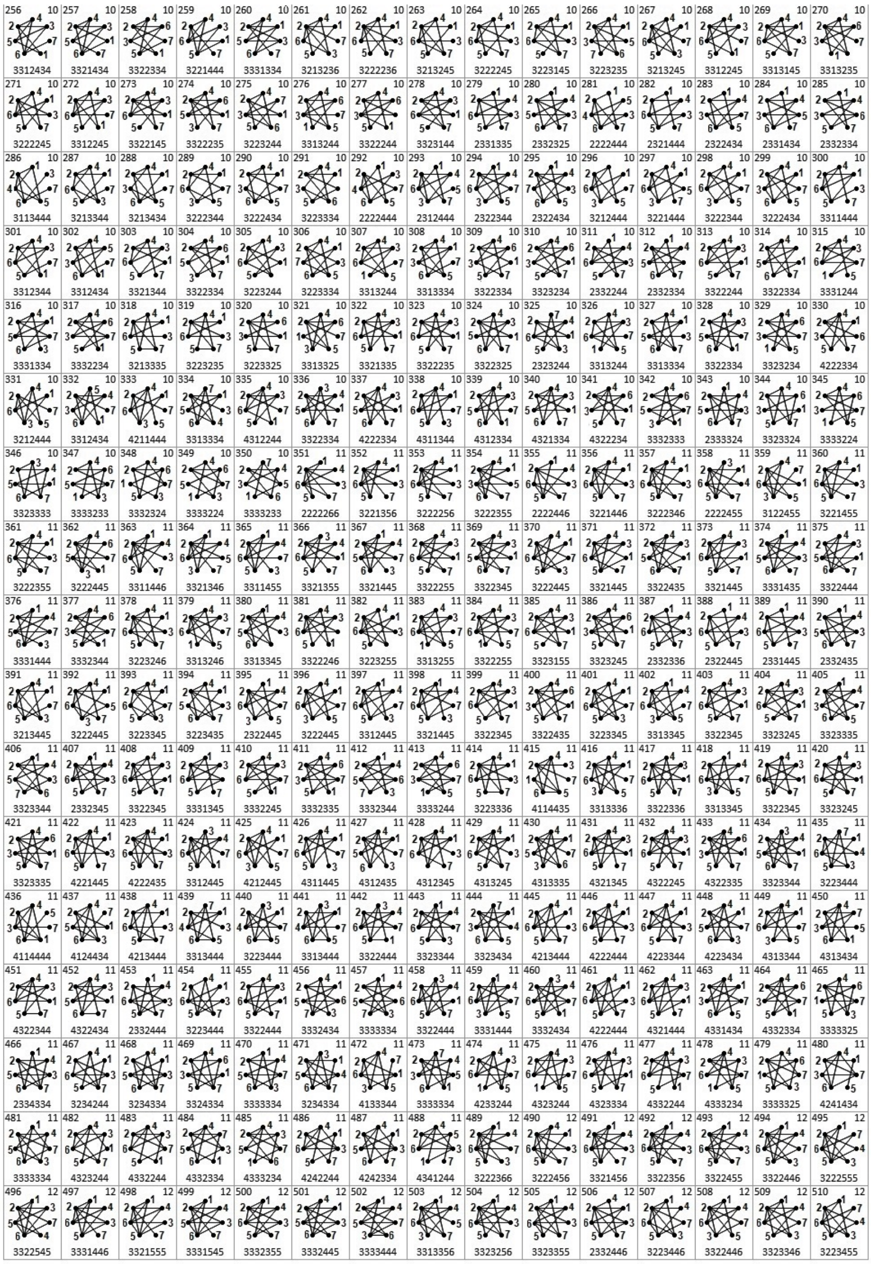

Figure A.2: A divisor cordial labeling for all connected graphs with seven vertices

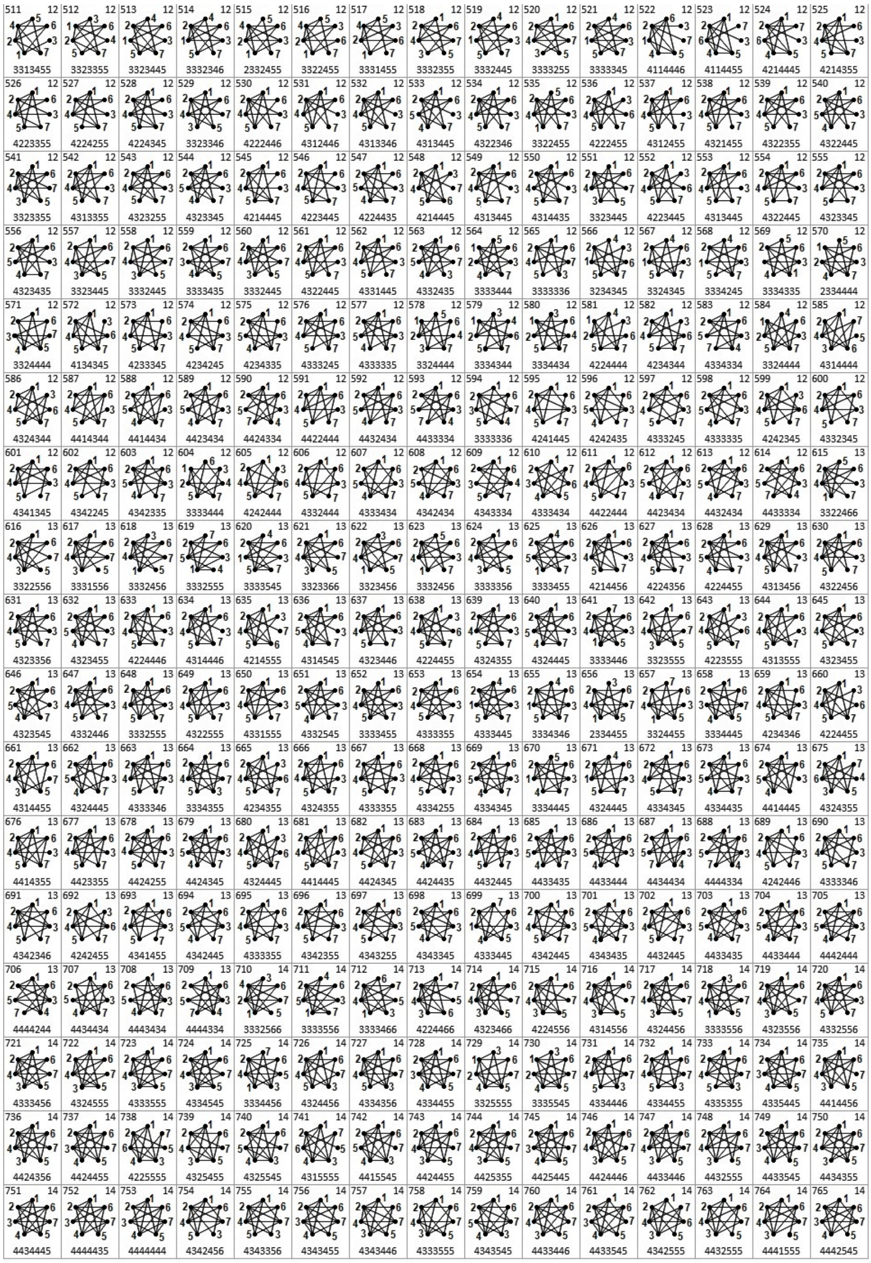

Figure A.3: A divisor cordial labeling for all connected graphs with seven vertices

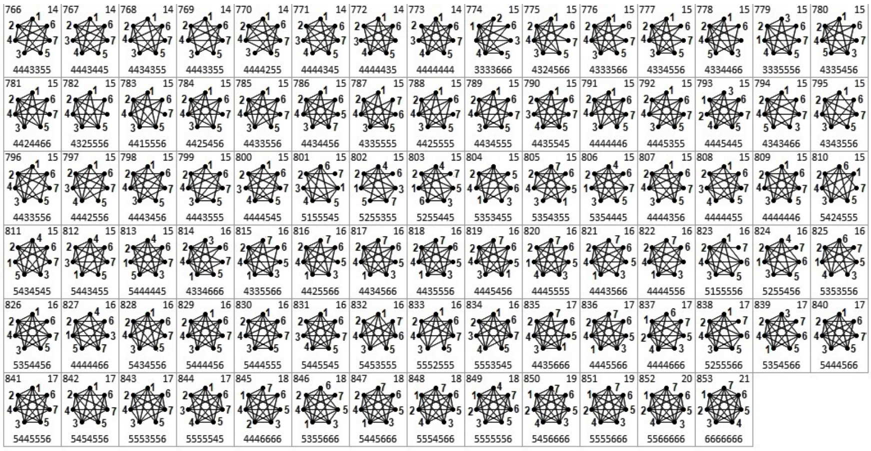

Figure A.4: A divisor cordial labeling for all connected graphs with seven vertices