

Ain Shams University

On Difference Cordial Graphs and

Other Graphs

Thesis by

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Submitted to

Department of mathematics - Faculty of Science

Ain Shams University - Cairo - Egypt

for the Degree of Doctor of Philosophy

in Pure Mathematics

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Assistant Professor of Pure Mathematics Department of Mathematics-Faculty of Science Ain Shams University Cairo 2016





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Shakir Al-Azzawy

Abstract

Graph labeling is one of the important branches of Graph Theory and became a principal tool in many applications on different sciences and technologies. All that leads to appearance of more than one type of labeling and multiple techniques to meet the required purposes.

In this thesis we study the two main types of graph labeling and introduce the labelings for interested families of graphs and a tractive results for graphs of these types. We state some basic definitions and theorems in graph theory which we need. We divide the other work into four chapters:

In chapter two we introduce some results in difference cordial graphs and difference cordial labelings for some families of graphs such as: ladder, triangular ladder, grid, step ladder and two sided step ladder graph. Also we discussed some families of graphs which may be difference cordial or not, such as diagonal ladder and some types of one-point union of graphs.

In chapter three we introduce some results on difference cordial graphs, where we present results concerning the relation between difference cordiality and the lengths of paths on graphs and study the Semi-Hamiltonian graph, biconnected outerplanar graphs and the line graph of a graph. Also, we describe the difference cordial labeling for some families of graphs such as: the graph obtained by duplication a vertex by an edge, bow graphs, butterfly graphs, shell-flower graphs and one-point union of complete graphs.

In chapter four we introduce some results on divisor cordial graphs and describe the divisor cordial labeling for the families of graphs: the jelly fish graph, the shell, the bow graph, butterfly graphs and the friendship graphs. In the last chapter we introduce results in divisor cordial labeling for regular graphs, divisor labelings for all graphs with number of vertices less than eight, and divisor cordial labelings for some types of trees such as: olive trees, spider trees, m-star trees, k-distant trees, caterpillar trees and banana trees.

Contents

A	CKN	OWLEDGMENT	vi
Al	bstra	let	vii
Li	st of	Figures	xii
Su	ımm	ary	1
Co	onter	nts	1
1 Introduction			4
	1.1	Brief Introduction to Labeling	4
	1.2	Some Fundamentals in Graph Theory:	8
		1.2.1 Some Types of Graphs	12
		1.2.2 Operations on Graphs	15
2	On	Difference Cordial Graphs	17
	2.1	Introduction	17
	2.2	Main Results	19
	2.3	Difference cordial labeling for Some graphs	25

		2.3.1	Ladder graphs L_n	25
		2.3.2	Triangular ladder graph TL_n	31
		2.3.3	The Grid graph $P_m \times P_n$	32
		2.3.4	Step ladder graph $S(T_n)$:	37
		2.3.5	Double Sided Step Ladder Graph $2S(T_{2n})$:	39
3	S	Some	Results and Examples on Difference Cordial	
	Gra	phs		51
	3.1	Intro	luction	51
	3.2	Some	Results	53
	3.3	Differ	ence Cordial Labeling for Some Families of Graphs .	59
		3.3.1	Graph Obtained by Duplication of Vertex by an Edge	59
		3.3.2	Bow Graphs	61
		3.3.3	Butterfly Graphs	62
		3.3.4	Shell-Flower Graphs	63
		3.3.5	One-Point Union of Complete Graphs	65
4	Son	ne Res	ults on Divisor Cordial Graphs	68
	4.1	Intro	luction	68
	4.2	The F	Results	70
	4.3	Diviso	or Cordial Labeling for Some Families of Graphs	72
		4.3.1	The Jelly Fish Graph	72
		4.3.2	The shell and The Bow Graph	74
		4.3.3	Butterfly Graphs	78
		4.3.4	Friendship Graphs	79

5 Divisor Cordial Labeling for Some Trees and Families						
	Gra	Graphs				
	5.1	Introd	uction	81		
	5.2	The R	esults	83		
	5.3	Diviso	r Cordial Labeling for Some Trees	85		
		5.3.1	Olive Tree	86		
		5.3.2	Spider Tree	88		
		5.3.3	m-stars Tree	88		
		5.3.4	k-distant tree	90		
		5.3.5	Caterpillar Tree	90		
		5.3.6	Banana Tree	91		

A All Nonisomorphic Graphs with 7 Vertices and its Divisor Cordial Graphs 98

List of Figures

1.1	The path and the cycle.	10
1.2	A two isomorphic graphs.	11
1.3	The bipartite graph.	12
1.4	Some operations on graphs	16
2.1	The graph $G = (8, 15)$	21
2.2	The graph $G = (9, 17)$	22
2.3	The graph $G = (6, 12)$	23
2.4	The flower graph Fl_8	24
2.5	The Ladder Graph L_n	26
2.6	Ladder Graph L_{10}	30
2.7	Ladder Graph L_{11}	31
2.8	Triangle Ladder Graph TL_n	31
2.9	A difference cordial labeling for TL_6	32
2.10	A difference cordial labeling for TL_7	32
2.11	The grid graph $P_m \times P_n$	33
2.12	A difference cordial labeling for grid graph $P_4 \times P_3$	36
2.13	A difference cordial labeling for grid graph $P_5 \times P_8$	36

2.14	The step ladder graph $S(T_n)$	37
2.15	Difference cordial labeling for the step ladder graph $S(T_{12})$	39
2.16	Double sided step ladder graph $2S(T_{10})$	40
2.17	The difference cordial labelings for the diagonal ladder	
	graphs DL_2 and DL_3	47
2.18	The graph $F_n^{(m)}$	48
2.19	The friendship graph F_5	50
3.1	A difference cordial labeling for the maximal outerplanar	
	graph with 12 vertices	57
3.2	An outerplanar graph with 12 vertices and 20 edges	57
3.3	Two disjoint difference cordial graphs.	59
3.4	The difference cordial labeling for the graph obtained by	
	duplication of vertex of C_7 by an edge	60
3.5	The bow graph with $m + n + 1$ vertices and its difference	
	cordial labeling.	62
3.6	A difference cordial labeling for the butterfly graphs	63
3.7	A shell-flower graph with k petals	64
3.8	A difference cordial labeling for the shell-flower graph with	
	two petals	65
3.9	The graph $K_5^{(2)}$	65
4.1	A Jelly fish graph $j(6, 11)$ and its divisor cordial labeling .	73
4.2	A shell graph $C(13, 10)$ and its divisor cordial labeling \ldots	75
4.3	A bow graph with $m = 13, n = 16$ and its divisor cordial	
	labeling	77

4.4	A divisor cordial labeling for the butterfly with 29 vertices	79
4.5	A friendship graph F_6 and its divisor cordial labeling	80
5.1	A divisor cordial labeling for all connected graphs with four	
	vertices except K_4	83
5.2	A divisor cordial labeling for all connected graphs with five	
	vertices.	83
5.3	A divisor cordial labeling for all connected graphs with six	
	vertices.	84
5.4	The olive tree ol_n	86
5.5	A divisor cordial labeling for olive tree ol_6	87
5.6	A divisor cordial labeling for spider tree	89
5.7	The 4–star graph and its divisor cordial labeling \ldots .	89
5.8	A divisor cordial labeling for the k -distant tree	90
5.9	Caterpillar tree	91
5.10	Banana tree	91
A.1	A divisor cordial labelin g for all connected graphs with	
	seven vertices	99
A.2	A divisor cordial labeling for all connected graphs with	
	seven vertices	100
A.3	A divisor cordial labeling for all connected graphs with	
	seven vertices	101
A.4	A divisor cordial labeling for all connected graphs with	
	seven vertices	102

Summary

This thesis sheds light on the two concepts of types of graph labeling and describe the labeling for many families of graphs.

Graph labeling is one of the famous problems in Graph Theory. Recently graph labeling became more important because the growth of its applications in many of sciences and technology on a different area such as: computer programming, coding theory, neural network, biotechnology, in the study of X-Ray crystallography, radar, communication network, circuit layouts. In this work by a graph G = (V, E), we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Harary [14] and for graph labeling, Gallian [12] is referred to.

In general Graph labeling is a strong communication between Number theory and structure of graphs. Nowadays nearly 200 graph labelings techniques have been studied.

Throughout this work we present new results in two types of graph labelings, and discuss the labeling of many kinds of graphs in chapters 2, 3, 4 and 5. In chapter two: the basic definitions and theorems of graph theory are introduced which are useful in current work, and the outline of the thesis.

In year 2013, Ponraj, Shathish Naraynan and Kala introduce the notions of difference cordial labeling for finite undirected and simple graph. In chapter two, we present some new results on difference cordial graphs under a title "On Difference Cordial Graphs" which are published in the mathematical Bulgarian journal "Mathematica AEterna" journal. This chapter comprises four sections and present new interesting results and facts in difference cordial graphs:

- Seven results concerning the degree of vertices and difference cordiality.
- One result concerning the graph and its complements.

In addition, we describe the function of labeling for different families of graphs such as: ladder, step ladder, two sided step ladder, diagonal ladder, triangular ladder, grid graph and some types of one-point union of graphs. All that appears in:

Mathematica Aeterna, Vol. 5, 2015, no. 1, 105 – 124. [37]

In **chapter three**: Some new results and examples on difference cordial graphs, and interested results about:

• five results about relation between the lengths of disjoint paths in graph and difference cordiality of a graph, in addition, Petersen, semi-Hamiltonian graph and outerplanar graph.

- Two results about line graph.
- A result for union of graphs.

Also we describe difference cordial labelings for the families of graphs: bow, buttery, Shell-Flower and One-Point Union of Complete graphs. These results are published in the academic journal "TURKISH JOURNAL OF MATHEMATICS" in Turkey.

Turk J Math, 40, (2016), 417-427 [38]

By combining the divisibility concept in Number theory and Cordial labeling concept in Graph labeling, Varatharajan, Navanaeethakrishnan Nagarajan in 2011, introduced a new concept called divisor cordial labeling.

In **chapter four**: some new results on divisor Cordial graph labeling are introduced:

new general results in divisor cordial labeling, four results in maximal number of edges are labeled one in any graph and in the regular graph. We introduce and discusses mappings of labelings for some families of graphs such as: jelly fish, shell, bow, butterfly and friendship graph, these results are submitted for publication in the Indian academic journal: Journal of Graph Labeling.

In **chapter five**: new results in divisor cordial labeling for the regular graphs, and divisor cordial labelings for all graphs with number of vertices less than eight except the graph K_4 and proof it not divisor cordial graph. As well the divisor cordial labeling for the trees: olive trees, spider trees, m-star trees, k-distant trees, caterpillar trees and banana trees.

Chapter 1

Introduction

1.1 Brief Introduction to Labeling

Graph labeling is a strong communication between Number Theory and structure of graphs. It is an assignment of integers to the vertices, edges, or both, subject to certain conditions. Most graph labeling methods extract their origin from a paper introduced firstly by Rosa in 1964 [30]. Diverse types are the subject of much study, where during the last 50 years nearly 200 graph labelings techniques have been studied in over 2000 papers [12].

Graph Labeling is a powerful tool that makes things ease in various fields of computer science, public key cryptography, Networks representation, database management [28].

Most of the graph labeling problems have three ingredients: A set of numbers S from which the labels are chosen; rule that assigns a value to each vertex or edge such that some conditions must be satisfied [12].

The problems related to labeling of graphs challenge our mind for their eventual solutions. Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes, convolution codes with optimal autocorrelation properties, X-ray crystallography, communication network, bio-technology and to determine optimal circuit layouts. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb [5].

A graceful labeling is an assignment of the integers $\{1, 2, ..., n\}$ to vertices of a graph such that once each edge is labeled with difference of its incident vertices, with each integer in $\{1, 2, ..., n - 1\}$ is used once and only once. In [30] Rosa has identified essentially three reasons why a graph fails to be graceful:

- 1. has "too many vertices" and "not enough edges",
- 2. has too many edges,
- 3. has the wrong parity.

Rosa [30] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Seoud and Abdel-Aal [32] determined all odd-graceful graphs of order at most 6 and proved that if G is odd-graceful then $G \cup K_{m,n}$ is oddgraceful. Seoud and Helmi in [33] proved: if G has an odd-graceful labeling f with bipartition (V_1, V_2) such that:

 $max \{ f(x) : f(x) \text{ is even}; \ x \in V_1 \} \ < \ min \{ f(x) : f(x) \text{ is odd}; \ x \in V_2 \},$

then G has an α -labeling, if G has an α -labeling, then $G \odot k_n$ is oddgraceful, and if G_1 has an α -labeling and G_2 is odd-graceful, then $G_1 \cup G_2$ is odd-graceful.

They also proved the following graphs have odd-graceful labelings: dragons obtained from an even cycle; graphs obtained from a gear graph by attaching a fixed number of pendent edges to each vertex of degree 2 on rim of the wheel of the graph; $C_{2m} \odot \bar{K_n}$; graphs obtained from an even cycle by attaching a fixed number of pendent edges to every other vertex; graphs obtained by identifying an endpoint of a star $S_n (n \ge 3)$ with a vertex of an even cycle; the graphs consisting of two even cycles of the same order that share a common vertex with any number of pendent edges attached at the common vertex; and the graphs obtained by joining two even cycles of the same order by an edge.

Seoud and Wilson [39] proved that $C_3 \bigcup K_4$, $C_3 \bigcup C_3 \bigcup K_4$ and certain graphs of the form $C_3 \bigcup P_n$ and $C_3 \bigcup C_3 \bigcup P_n$ are not graceful. Seoud and Youssef [40] investigated the gracefulness of specific families of the form $G \bigcup K_{m,n}$. They obtained the following results: $C_3 \bigcup K_{m,n}$ is graceful if and only if $m \ge 2$ and $n \ge 2$; $C_4 \bigcup K_{m,n}$ is graceful if and only if $(m,n) \ne (1,1); C_7 \bigcup K_{m,n}$ and $C_8 \bigcup K_{m,n}$ are graceful for all m and n; $mK_3 \bigcup nK_{1,r}$ is not graceful for all m, n and $r; K_i \bigcup K_{m,n}$ is graceful for $i \le 4$ and $m \ge 2; n \ge 2$ except for i = 2 and $(m,n) = (2,2); K_5 \bigcup K_{1,n}$ is graceful for all $n; K_6 \bigcup K_{1,n}$ is graceful if and only if n is neither 1 nor 3.

Another best known labeling methods are called harmonious labelings. The harmonious graphs naturally arose in the study by the two researchers Graham and Sloane [13]. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers (modulo q) such that when each edge xy is assigned the label $f(x) + f(y) \pmod{q}$, the parity condition for harmonious graph the resulting edge labels are distinct.

Seoud and Elsakhawi [36] proved: paths and ladders are arbitrarily graceful; and for $n \ge 3$; K_n is k-graceful if and only if k = 1 and n = 3 or 4.

Seoud and Youssef [41] have shown that the one point union of a triangle and C_n is harmonious if and only if $n \equiv 1 \pmod{4}$ and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. They [35] introduced Gracefulness of the union of cycles and paths. Also in [34] Seoud, Abdel Maqsoud, and Sheehan noted that when r or s is even, rC_8 is not harmonious. They proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ are not harmonious when m+n is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has an odd order.

Cordial labeling is a variation of both graceful and harmonious labelings introduced by I. Cahit in 1987 [8].

A-cordial labelings defined as a common generalization of cordial labeling (introduced by Cahit [8]) and harmonious labeling (introduced by Graham and Sloane [13]).

Ponraj, Sathish Narayanan, and Kala [23] introduced the notion of difference cordial labelings.

By combining the divisibility concept in number theory and cordial labeling concept in graph labeling, R. Varatharajan, S. Navanaeethakrishnan, and K. Nagarajan introduce a new concept called divisor cordial labeling [46].

We study and present new results on the last two types of labelings mentioned above. All graphs in this work are simple, finite and undirected.

Graph labelings of diverse types are the subject of much study and the state of the field is described in detail in Gallian dynamic survey [12]. The results obtained so far, while numerous, are mainly piecemeal in nature and lack generality. In an attempt to provide something of a framework for these results, we introduce some of them in the next chapter.

1.2 Some Fundamentals in Graph Theory:

In this section we will describe some of graphs and its properties that we need in our work. Since the language of graph theory is still not standard, all authors have their own terminology.

A graph G consists of a non-empty finite set V(G) of elements called vertices, and a finite family E(G) of unordered pairs of (not necessarily distinct) elements of V(G) called edges; the use of the word 'family' permits the existence of multiple edges. We call V(G) the vertex set and E(G) the edge family of G. Although we sometimes have to restrict our attention to simple graphs (in any simple graph there is at most one edge joining a given pair of vertices), we shall prove our results for general graphs whenever possible. An edge v, w is said to join the vertices v and w, and is again abbreviated to vw. The number of vertices in G is called the order of G and the number of edges in G is called the size of G. The order and size of G are denoted by p and q respectively, in other word |V(G)| = p and |E(G)| = q. A graph is trivial if its vertex set is a singleton.

Let G = (V, E) be a graph. Two vertices v_1 and v_2 are said to be adjacent if there exists an edge $e \in E$, $e = v_1, v_2$; $v \in V$. Two edges e_1 and e_2 are said to be adjacent if there exists a common vertex v on them.

Let $v \in V$, G = (V, E). The neighbors of v are the set of vertices that are adjacent to v. Formally: $N(v) = \{u \in V : e \in E, e = u, v\}.$

The degree of a vertex v of a graph G is the number of edges incident to the vertex, with loops counted twice. It is denoted by deg(v), that means deg(v) = |N(v)|. The degree sequence of a graph is the sequence formed by arranging the vertex degrees in non - increasing order.

A vertex of degree zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex or a leaf. An edge e in a graph Gis called a pendant edge if it is incident with a pendant vertex. Note: in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum. This result is called the handshaking lemma.

A graph G = (V, E) is a simple graph if G has no edges that are self-loops and the set E(G) consists of distinct unordered pairs of distinct elements of V(G). Thus every simple graph is a graph, but not every graph is a simple graph, we will assume that every graph we discuss in these notes is a simple graph and we will use the term graph to mean simple graph. When a particular result holds in a more general setting, we will state it explicitly.

A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is a proper subgraph of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A spanning subgraph of Gis a subgraph H of G with V(H) = V(G).

Much of graph theory involves 'walks' of various kinds. A walk is a sequence of vertices and edges $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_k$, in which each edge $e_i = v_{i-1}v_i$. This walk goes from v_0 to v_k or "connect" v_0 with v_k , the length of walk is the number of its edges, and if $v_0 = v_k$ the walk is closed. The important types of walk are: the path is a walk, the trail is a walk in which no edge is repeated, and the cycle is a non trivial closed trail in which no vertex is repeated. Usually the path with n vertices is denoted by P_n and the cycle with n vertices by C_n , the least cycle (when n = 3) is called triangle. The path P_6 and cycle C_6 are shown in Figure 1.1.



Figure 1.1: The path and the cycle.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f: V_1 \longrightarrow V_2$ that preserves the adjacency, i.e., $uv \in E_1$, if and only if, $f(u)f(v) \in E_2$. The function f is then called an isomorphism between G_1 and G_2 . If G_1 and G_2 are isomorphic, then we write $G_1 \cong G_2$, the two graphs G_1 and G_2 shown in Figure 1.2 are isomorphic.



Figure 1.2: A two isomorphic graphs.

Clearly "if two graphs are isomorphic then they have, same number of vertices, same number of edges and umber of vertices" having same degree is equal.

A graph is said to be connected if for every pair of distinct vertices u, v there is a u, \dots, v path joining them. A graph that is not connected is called disconnected, a maximal connected subgraph of a disconnected graph is called a component of the graph and every connected graph has exactly one component, in other words, a graph is connected if it cannot be expressed as the union of two graphs, and disconnected otherwise. Each one of the two graphs G_1 and G_2 shown in Figure 1.2 is connected.

1.2.1 Some Types of Graphs

This subsection presents the definitions for some types of graphs that we may remember or study in this work.

A Null graph is a graph with n vertices and has no edge. A graph in which all the vertices have equal degree is called a regular graph. If for every vertex v of graph G, d(v) = k for some $k \in \mathbb{N}$, then G is k-regular graph [14]. The null graph is 0 - regular graph and every cycle C_n is a 2-regular graph with n vertices.

A complete graph G = (V, E) on n vertices has n vertices and for each pair of vertices $u, v; uv \in E(G)$. That means in the complete graph with n vertices every two of which are adjacent, then: $|E(G)| = \frac{1}{2}n(n-1)$. A complete graph on n vertices is denoted by K_n . Note that K_n is (n-1) - regular and the null graph with one vertex is K_1 .

A graph G = (V, E) is said to be bipartite if the vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that for every edge $e_i = v_i v_j \in E$, $v_i \in V_1$ and $v_j \in V_2$. Figure 1.3 shows the bipartite graph. Notice the vertices set of bipartite graph contain at least two vertices and at least one edge.



Figure 1.3: The bipartite graph.

We can also define n - partite graphs as: A graph G = (V, E) is called n - partite graph if the vertex set V can be partitioned into n nonempty sets $V_1, V_2, ..., V_n$ such that every edge of G joins the vertices from different subsets. It is often called a *multipartite* graph.

The complete bipartite graph $K_{m,n}$ is the bipartite graph whose vertex set is partitioned into two non-empty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$, and any vertex in V_1 is adjacent to each vertex in V_2 , and any two distinct vertices in V_i are not adjacent to each other. The number of edges in complete bipartite graph $K_{m,n}$ is mn. If m = n, then $K_{n,n}$ is n - regular. When m = 1 then $K_{1,n}$ is called the star graph.

An n-partite graph G is called complete n-partite if for each $i \neq j$, each vertex of the subset V_i is adjacent to every vertex of the subset V_j . A complete n-partite graph with n partitions of vertex set is denoted by K_{m_1,m_2,\ldots,m_n} .

A graph is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no any two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar graph. A simple planar graph is called maximal planar if no edge can be added without destroying its planarity [6].

A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same "face". An outerplanar graph is maximal if no edge can be added without losing outerplanarity [6].

A graph which does not contain any cycle is known an acyclic graph and if it includes exactly one cycle it is called a unicyclic graph [4]. A forest is a graph that contains no cycles, and a connected forest is a tree [48]. For a tree T of order n, T is connected, has no cycles, and has n-1 edges [48]. A star graph with n vertices is a tree with one vertex having degree n-1 and other n-1 vertices having degree 1 and denoted by S_{n-1} . There are many types of trees we will discuss some of them in chapter five.

A cycle passing through all the vertices of a graph is called Hamiltonian graphs. A graph containing a Hamiltonian cycle is called Hamiltonian graphs. A path passing through all the vertices of a graph is called Hamiltonian path and a graph containing Hamiltonian path is said to be semi-Hamiltonian.

Define the general form of Petersen graph as: the graph P(k,m) = (V,E) where $V = \{u_i, v_i : i = 1, 2, ..., k-1\}$ and $E = \{u_i u_{i+l}, v_i v_{i+m}, u_i v_i : i = 0, 1, ..., k-1\}$ where addition is modulo k and $m < \frac{1}{2} k$ [15]. The Petersen graph P(5,2) which shown as G_1 in Figure 1.2 consider the stander Petersen graph.

For a graph G = (V, E) define the line graph, denoted by L(G), as the graph with vertices consisting of the edges of G, that is V(L(G)) = E(G), and where $e, e \in V(L(G))$ are adjacent in L(G) if, and only if, they are adjacent in G [2].

Let G = (V, E) be a graph, the graph complement of G is the graph $G^c = (V, E^c)$ so that: $E^c = \{uv : u, v \in V \text{ and } u \neq v \text{ and } uv \notin E\}$ [2]. A graph is said to be self complement if $G \cong G^c$. The complement of complete graph K_n is the null graph with n vertices. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. It is said that G_1 and G_2 are disjoint if they have no vertex in common and edge disjoint if they have no edge in common.

1.2.2 Operations on Graphs

There are several ways to get new graphs from old. Briefly described some of these operations in this subsection:

The disconnected graph G = (V, E), where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$ [14]. The union of k graphs isomorphic to G is denoted by kG [4]. The one point union of t cycles, each of length n is denoted by C(t) n is called the friendship graph.

If $V1 \cap V2 = V$, then the graph G = (V, E), where $V = V1 \cap V2$ and $E = E1 \cap E2$ is called the intersection of G_1 and G_2 and is denoted by $G_1 \cap G_2$.

If G_1 and G_2 are disjoint graphs, then the join of G_1 and G_2 is denoted by $G_1 + G_2$ and is defined as $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$

The Cartesian product $G \times H$ has vertex-set $V(G) \times V(H)$, and (v_i, w_j) is adjacent to (v_h, w_k) if either v_i is adjacent to v_h in G and $w_j = w_k$, or $v_i = v_h$ and w_j is adjacent to w_k in H. The ladder graph L_n is the Cartesian product of the paths P_n and P_2 , i.e. $L_n = P_n \Box P_2$. Figure 1.4 shows some operation on graphs.



Figure 1.4: Some operations on graphs

The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (of V_1 vertices) and V_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2 .

Chapter 2

On Difference Cordial Graphs

In this chapter we introduce some results in difference cordial graphs and the difference cordial labeling for some families of graphs as: ladder, triangular ladder, grid, step ladder and two sided step ladder graph. Also we discussed some families of graphs which may be difference cordial or not, such as diagonal ladder graphs and some types of one-point union of graphs.

2.1 Introduction

In this chapter we will deal with finite, simple and undirected graphs. By the expression G = (V, E) we mean a simple undirected graph with vertex set V, |V| is called the order of graph and edge set E, |E| is called its size. Graph labeling connects many branches of mathematics and is considered one of important blocks of graph theory, for more details see [14]. Cordial labeling was first introduced in 1987 by Cahit [8], then there was a major effort in this area made this topic growing steadily and widely, see [12].

In [23] Ponraj, Shathish Naraynan and Kala introduce the notions of difference cordial labeling for finite undirected and simple graph, as in the following definition:

Definition 1. Let G = (V, E) be a (p,q) graph, and f be a map from V(G) to 1, 2, ..., p. For each edge uv assign the label |f(v) - f(u)|, f is called a difference cordial labeling if f is one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ denotes the number of edges labeled with 1 while $e_f(0)$ denotes the number of edges not labeled with 1. A graph with a difference cordial labeling is called a difference cordial graph [23].

Ponraj et al. show every graph is a subgraph of a difference cordial graph and any r-regular graph with $r \ge 4$ is not difference cordial graph, every path and cycle are difference cordial graphs, the star graph $K_{1,n}$ is difference cordial if and only if $n \le 5$, the graph K_n is difference cordial only when $n \le 4$ while the bipartite graph $K_{m,n}$ is not difference cordial if $m \ge 4$ and $n \ge 4$, the bistar $B_{m,n}$ is not difference cordial when $m+n \ge 9$ but the wheel W_n , the fan F_n , the gear G_n , the helm H_n and all webs are difference cordial graphs for all n [23].

In [24] the authors investigated the difference cordial labeling behavior of $G \odot P_n$, $G \odot mK_1$ (m = 1, 2, 3) where G is either unicyclic or a tree and $G_1 \odot G_2$ are some more standard graphs. Some graphs obtained from triangular snake and quadrilateral snake were investigated with respect to the difference cordial labeling behavior. Also the behavior of subdivision of some snake graphs is investigated in [24].

Proposition 2.1. If G is a (p,q) difference cordial graph, then $q \leq 2p - 1$ [23].

Definition 2. The number $\delta(G) = \min \{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max \{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, the number $d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$ is the average degree of the vertices in the graph G [10].

Definition 3. A fan graph is obtained by joining all vertices of a path P_n to a further vertex, called the center. Thus F_n contains n + 1 vertices say $c, v_1, v_2, v_3, ..., v_n$ and 2n - 1 edges, say $cv_i, 1 \leq i \leq n$, and, $v_iv_{i+1}, 1 \leq i \leq n - 1$.

Notation 2.2. The maximum number of edges labeled 1, that is related with a specific vertex, equals 2.

2.2 Main Results

Proposition 2.3. The graph G(p,q) is not difference cordial graph if $\delta(G) \ge 4$.

Proof. Let G(p,q) be any graph with $\delta(G) \ge 4$; then, the minimum value of q is 2p; but 2p > 2p - 1, this contradicts proposition 2.1.

Proposition 2.4. The graph G(p,q) is not difference cordial if $d(G) \ge 4$.

Proof. Let G(p,q) be any graph with $d(G) \ge 4$; then the value of q is more than or equal to 2p, but $2p \le 2p - 1$, which is contradicts Proposition 2.1.

Remark 2.5. The value of $e_f(0)$ is not exceeding p in any difference cordial graph G(p,q).

Proof. Direct consequence of Proposition 2.1.

Proposition 2.6. Let G(p,q) be a graph with two vertices of degree (p-1), then G is not difference cordial for all $p \ge 8$.

Proof. Let G(p,q) be a graph with p vertices, $p \ge 8$ and has two vertices v_i, v_j of degree (p-1) then there are 2p-3 different edges incident with them, If there are more than two additional edges then G is not difference cordial since $q \le 2p-1$. If there are only two additional edges then q = 2p-1, then we have two cases:

Case 1: the edge connecting v_i and v_j is labeled 0, then there are at most 6 edges are labeled 1: two passing through v_i , two are passing through v_j and the two additional edges.

in this case

$$|2p - 7 - 6| = |2p - 13| \ge 2$$
 where $p \ge 8$

i.e., G is not difference cordial.

Case 2: the edge connecting v_i and v_j are labeled 1, then there are at most 5 edges labeled 1: one passing through v_i and v_j , two edges are: one

is incident with v_i and other is incident with v_j and the two additional edges. In this case

$$|2p - 6 - 5| = |2p - 11| \ge 2$$
 where $p \ge 7$

i.e., G is not difference cordial.

In case there is one additional edge, other than those incident with v_i, v_j , similar argument is used.

Example 2.7. $\deg(V_8) = 7$, $\deg(V_7) = 7$



Figure 2.1: The graph G = (8, 15)

notice: G cannot be a difference cordial graph.

Proposition 2.8. Let G(p,q) be any graph with two vertices of degrees (p-1) and (p-2); then G is not a difference cordial graph for all $p \ge 9$.

Proof. Similar to the proof of Proposition 2.6.

Example 2.9. In Figure 2.2

 $\deg(V_8) = 7$, $\deg(V_9) = 8$, G cannot be a difference cordial graph.



Figure 2.2: The graph G = (9, 17)

In [25] theorem 2.14, R. Ponraj, S. Sathish Narayanan and R. Kala state that, Let G be a (p,q) difference cordial graph with k(k > 1) vertices of degree p - 1. Then $p \le 7$. However:

Corollary 2.10. The graph G(p,q) is not a difference cordial graph if there exist three vertices of degree (p-1) for all $p \ge 6$.

Proof. Let G(p,q) be a graph with three of its vertices of degree p-1 then there exist at least 3p-6 edges in the graph, by proposition 2.1 if the graph is a difference cordial graph then

$$3p - 6 \le 2p - 1$$

A contradiction when $p \ge 6$.

Example 2.11. In Figure 2.3

 $12 \not\leq 2 * 6 - 1$ and G cannot be a difference cordial graph.



Figure 2.3: The graph G = (6, 12)

Proposition 2.12. Let G be a (p,q) graph with one vertex of degree (p-1) then G is not a difference cordial if there exists a set of non adjacent vertices S with $\sum_{v_i \in S} (\deg(v_i) - 3) \ge 4$.

Proof. Let G be a (p,q) graph with p vertices and have a vertex v_k of degree p-1 and there exists a set of non adjacent vertices S with $\sum_{v_i \in S} (\deg(v_i) - 3) \ge 4$. Then there are at least p-3 edges passing through v_k are labeled 0, hence $e_f(0) \ge p-3+4 = p+1$, i.e., G is not a difference cordial graph.

Example 2.13. In the Figure 2.4 n = 17, q = 32, deg(v) = 16

 $S = \{v_1, v_3, v_5, v_7\} then$ $\sum_{v_i \in S} (\deg(v_i) - 3) = 1 + 1 + 1 + 1 = 4$

there are at least 4 + 14 = 18 edges labeled 0, then the graph is not difference cordial.


Figure 2.4: The flower graph Fl_8

Proposition 2.14. Let G be a (p,q) graph then G is not difference cordial graph if there exists a set of non adjacent vertices S with $\sum_{v \in S} (\deg(v) - 2) = p + 1.$

Proof. Let Sbe of vertices a set non adjacent with $\sum_{v_i \in S} (\deg(v_i) - 2) = p + 1$. Since the maximum number of edges labeled 1 that are incident with a specific vertex equals 2, then the number of edges labeled 0 that are incident with vertices of S are at least $\sum_{v_i \in S} (\deg(v_i) - 2)$ this means the minimum value for $e_f(0)$ in the graph G is p+1, therefor the graph cannot be difference cordial.

Proposition 2.15. The complement graph of a difference cordial graph is not difference cordial when the number of its vertices is more than eight.

Proof. Let G be a (p,q) difference cordial graph with $p \ge 9$, then by Proposition 2.6 :

$$q \le 2p - 1 \tag{2.1}$$

 G^c , the complement of graph G contains $\frac{1}{2}p(p-1) - q$ edges and p vertices, let G^c be difference cordial then

$$\frac{1}{2}p(p-1) - q \le 2p - 1 \tag{2.2}$$

by adding equation 2.1 and equation 2.2 we get

$$\frac{1}{2}p(p-1) \le 4p - 2p^2 - 9p \le -4$$

A contradiction for all $p \ge 9$

2.3 Difference cordial labeling for Some graphs

In This section we will discuss the ability of applying difference cordial labeling for some graphs and the functions which make it difference cordial graphs.

The Proposition 2.1 consider necessary condition for difference cordial labeling but it is not sufficient.

2.3.1 Ladder graphs L_n

The ladder graph is a planner undirected graph denoted by L_n with 2n vertices and 3n - 2 edges [14]. The ladder graph L_n can be expressed as $L_n \cong P_n \times P_2$, the Figure 2.5 show the ladder graph L_n



Figure 2.5: The Ladder Graph L_n

Proposition 2.16. Every ladder graph L_n is difference cordial for all n.

Proof. Let L_n be a ladder graph, then it has 2n vertices and 3n-2 edges. Let the vertices be $v_1, v_2, ..., v_{2n}$ such that $v_n v_{n+1}$ is an edge in this graph. Define the mapping $f : L_n \longrightarrow \{1, 2, ..., 2n\}$ by:

$$f(v_i) = \begin{cases} i & if \quad 1 \le i \le \left\lceil \frac{1}{2} \left| E \right| \right\rceil \\ 3\left\lceil \frac{1}{2} \left| E \right| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2i & if \quad \left\lceil \frac{1}{2} \left| E \right| \right\rceil < i \le \left\lceil \frac{1}{2} \left| E \right| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil \\ 2(i-n) - 1 & if \quad \left\lceil \frac{1}{2} \left| E \right| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil < i \le 2n \text{ and } n \text{ is odd} \\ 2(i-n) & if \quad \left\lceil \frac{1}{2} \left| E \right| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil < i \le 2n \text{ and } n \text{ is even} \end{cases}$$

From the first part of definition notice that there are $\left\lceil \frac{1}{2} |E| \right\rceil - 1$ of edges are labelled 1, in the second part we notice that:

$$|f(v_{i+1}) - f(v_i)| = \left| 3\left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2(i+1) - 3\left\lceil \frac{1}{2} |E| \right\rceil - \left\lceil \frac{1}{2}n \right\rceil - 2 + 2i \right|$$

= 2

 \mathbf{SO}

$$\begin{aligned} \left| f(v_i) - f(v_{2n-(i+1)}) \right| &= \left| 3 \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2i - 2n + (i+1) \right| \\ &= \left| 3 \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 3 - i - 2n \right| \\ &= \left| 3 \left\lceil \frac{1}{2}(3n-2) \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 3 - i - 2n \right| \\ &= \left| 3n - i \right| > 1 \end{aligned}$$

which means all these edges are labelled 0.

In the third part of definition we notice when n is even:

$$|f(v_{i+1}) - f(v_i)| = |2(i+1-n) - 2(i-n)| = 2$$

and

$$|f(v_i) - f(v_{2n-(i+1)})| = |2(i-n) - 2n + (i+1)| = |3i - 4n|$$

$$> |3(\left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil) - 4i|$$

$$> |3(\left\lceil \frac{1}{2}(3n-2) \right\rceil + \left\lceil \frac{1}{4}n \right\rceil) - 4n|$$

$$> \left| \frac{1}{2}n + 3\left\lceil \frac{1}{4}n \right\rceil - 3 \right|$$

$$> \begin{cases} |5m-3| & if \quad n = 4m \\ |5m+1| & if \quad n = 4m \end{cases}$$

$$> 2$$

this means all the edges $v_i v_{2n-(i+1)}$ in this third part are labeled 0.

But if n is an even number then the number of the total edges of the ladder L_n is even and thus there must exist additional edge labeled 1, which we may get it from the label of the last vertex in part two and the first label in part three.

Notice that if $i = \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil$ then

$$f(v_{\left\lceil\frac{1}{2}|E|\right\rceil+\left\lceil\frac{1}{4}n\right\rceil}) = 3\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{2}n\right\rceil + 2 - 2\left(\left\lceil\frac{1}{2}|E|\right\rceil + \left\lceil\frac{1}{4}n\right\rceil\right)$$
(2.3)

and if $i = \left\lceil \frac{1}{2} |E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1$, then

$$f(v_{\lceil \frac{1}{2}|E|\rceil+\lceil \frac{1}{4}n\rceil+1}) = 2\left(\left(\left\lceil \frac{1}{2}|E|\right\rceil + \left\lceil \frac{1}{4}n\right\rceil + 1\right) - n\right)$$
(2.4)

by subtracting (2.4) from (2.3) we get:

$$\begin{aligned} &f(v_{\lceil \frac{1}{2}|E|\rceil + \lceil \frac{1}{4}n\rceil}) - f(v_{\lceil \frac{1}{2}|E|\rceil + \lceil \frac{1}{4}n\rceil + 1}) \\ &= 3\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil\right) - 2\left(\left(\left\lceil \frac{1}{2}|E| \right\rceil + \left\lceil \frac{1}{4}n \right\rceil + 1\right) - n\right) \right) \\ &= -\left\lceil \frac{1}{2}|E| \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = -\left\lceil \frac{1}{2}(3n-2) \right\rceil + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil \\ &= \frac{-3}{2}n + 1 + \frac{5}{2}n - 4\left\lceil \frac{1}{4}n \right\rceil = n + 1 - 4\left\lceil \frac{1}{4}n \right\rceil = \left\{ \begin{array}{c} 1 & if \ n = 4m \\ -1 & if \ n = 4m + 2 \end{array} \right\} \end{aligned}$$

thus the edge $v_{\lceil \frac{1}{2}|E|\rceil+\lceil \frac{1}{4}n\rceil}v_{\lceil \frac{1}{2}|E|\rceil+\lceil \frac{1}{4}n\rceil+1}$ is labelled 1, then the graph is difference cordial.

Now if n is an odd number then |E| is an odd number and then from the first part we get $\lfloor \frac{1}{2} |E| \rfloor$ edges are labeled 1 and all other edges in the second and third part are labelled 0, similarly when n is even, and

$$\begin{split} f(v_{\lceil \frac{1}{2}|E|\rceil + \lceil \frac{1}{4}n\rceil}) &- f(v_{\lceil \frac{1}{2}|E|\rceil + \lceil \frac{1}{4}n\rceil + 1}) \\ &= 3\left\lceil \frac{1}{2}|E|\right\rceil + \left\lceil \frac{1}{2}n\right\rceil + 2 - 2\left(\left\lceil \frac{1}{2}|E|\right\rceil + \left\lceil \frac{1}{4}n\right\rceil\right) - 2\left(\left(\left\lceil \frac{1}{2}|E|\right\rceil + \left\lceil \frac{1}{4}n\right\rceil + 1\right) - n\right) + 1 \\ &= -\left\lceil \frac{1}{2}|E|\right\rceil + \left\lceil \frac{1}{2}n\right\rceil - 4\left\lceil \frac{1}{4}n\right\rceil + 2n - 1 \\ &= -\left\lceil \frac{1}{2}(3n-2)\right\rceil + \left\lceil \frac{1}{2}n\right\rceil - 4\left\lceil \frac{1}{4}n\right\rceil + 2n - 1 \end{split}$$

$$= \left\{ \begin{array}{l} -\left\lceil \frac{1}{2}(3(4m+1)-2)\right\rceil + \left\lceil \frac{1}{2}(4m+1)\right\rceil \\ -4\left\lceil \frac{1}{4}(4m+1)\right\rceil + 2(4m+1) - 1 & ifn = 4m+1 \\ \\ -\left\lceil \frac{1}{2}(3(4m+3)-2)\right\rceil + \left\lceil \frac{1}{2}(4m+3)\right\rceil \\ -4\left\lceil \frac{1}{4}(4m+3)\right\rceil + 2(4m+3) - 1 & ifn = 4m+3 \end{array} \right\}$$
$$= \left\{ \begin{array}{l} 0 & if \quad n = 4m+1 \\ 1 & if \quad n = 4m+3 \end{array} \right\}$$

then

$$e_f(1) = e_f(0)$$
 if *n* is even
 $e_f(1) = e_f(0) - 1$ if *n* is odd & $n = 4m + 1$
 $e_f(1) = e_f(0) + 1$ if *n* is odd & $n = 4m + 3$

Hence G is difference cordial.

Example 2.17. Consider the graph L_{10}

 $n = 10, |E| = 28, \ \left\lceil \frac{1}{2} |E| \right\rceil = 14, \ \left\lceil \frac{1}{2}n \right\rceil = 5, \left\lceil \frac{1}{4}n \right\rceil = 3 \ then:$

$$f(v_i) = \begin{cases} i & if \quad 1 \le i \le 14 \\ 49 - 2i & if \quad 14 < i \le 17 \\ 2(i - 10) & if \quad 17 < i \le 20 \end{cases}$$

 $f(v_1) = 1, \quad f(v_2) = 2, \quad \cdots, \quad f(v_{14}) = 14, \quad f(v_{15}) = 19,$ $f(v_{16}) = 17, \quad f(v_{17}) = 15, \quad f(v_{18}) = 16, \quad f(v_{19}) = 18, \quad f(v_{20}) = 20.$ $\underbrace{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10}_{19 + 2 + 13 + 10 + 13 + 12 + 11}$

Figure 2.6: Ladder Graph L_{10}

 $e_f(0) = 14, \quad e_f(1) = 14$

Example 2.18. Consider the graph L_{11}

$$n = 11, |E| = 31, \left\lceil \frac{1}{2} |E| \right\rceil = 16, \left\lceil \frac{1}{2}n \right\rceil = 6, \left\lceil \frac{1}{4}n \right\rceil = 3,$$

then

$$f(v_i) = \begin{cases} i & if \quad 1 \le i \le 16 \\ 56 - 2i & if \quad 16 < i \le 19 \\ 2(i - n) - 1 & if \quad 19 < i \le 22 \end{cases}$$

 $f(v_1) = 1, f(v_2) = 2, \dots, f(v_{16}) = 16, f(v_{17}) = 22,$ $f(v_{18}) = 20, f(v_{19}) = 18, f(v_{20}) = 17, f(v_{21}) = 19, f(v_{22}) = 21.$



Figure 2.7: Ladder Graph L_{11}

 $e_f(0) = 15, \ e_f(1) = 16$

2.3.2 Triangular ladder graph TL_n

A triangular ladder TL_n , $n \ge 2$, is a graph obtained from the ladder L_n = $P_n \times P_2$ by adding the edges $u_i v_{i+1} for 1 \le i \le n-1$, such graph has 2n vertices with 4n-3 edges, the triangular ladder graph TL_n is shown in the Figure 2.8.



Figure 2.8: Triangle Ladder Graph TL_n

Proposition 2.19. The triangular ladder graphs TL_n , $n \ge 2$ are difference cordial graph for all n.

Proof. Let $G = TL_n, n \ge 2$ be a triangular ladder graph, then G = (2n, 4n - 3). Define the function

$$f(v_i) = 2i - 1 \text{ and } f(u_i) = 2i, \ 1 \le i \le n$$
 (2.5)

It is clear that $e_f(1) = 2n - 1$ hence $e_f(0) = (4n - 3) - (2n - 1) = 2n - 2$, then $|e_f(0) - e_f(0)| = 1$, thus $G = TL_n, n \ge 2$ are difference cordial graphs for all n.

Example 2.20. Consider the graphs TL_6 and TL_7



Figure 2.9: A difference cordial labeling for TL_6



Figure 2.10: A difference cordial labeling for TL_7

2.3.3 The Grid graph $P_m \times P_n$

In this subsection we will investigate the difference cordial labeling for every grid graph of the form $P_m \times P_n$ for all m, n. Let the vertices of the grid graph be arranged as a sequence in certain order as in the Figure 2.11

This kind of graphs contains mn vertices and 2mn - (m+n) edges.



Figure 2.11: The grid graph $P_m \times P_n$

Proposition 2.21. Every grid graph is $P_m \times P_n$ is a difference cordial graph for all integers m, n > 1.

Proof. Let G be a graph $P_m \times P_n$ then G = (mn, 2mn - (m+n))Case 1: If m = n

then $|V| = n^2$ and $|E| = 2(n^2 - n)$, define the function f for labeling vertices of G by:

$$f(v_{ij}) = (i-1)n + j$$

in each row of the grid graph there exist n-1 edges labelled 1 this leads to $e_f(1) = n(n-1)$ and the number of edges labelled 0 is equal to:

$$2n(n-1) - n(n-1) = n(n-1),$$

thus G is a difference cordial graph.

Case 2: If |m - n| = 1, then |V| = mn and |E| = 2mn - (m + n). Let n = m + 1 then $|E| = 2m^2 - 1$.

Now using the same functions in **Case 1** we will get

$$e_f(1) = m(n-1) = m(m+1-1) = m^2$$

and

$$e_f(0) = (m-1)(m+1) = m^2 - 1$$

which means the graph is a difference cordial graph. Similarly if m = n + 1

Case 3: If $|m - n| \ge 2$.

Let n > m and let $k = \left\lceil \frac{1}{2}(n-m) \right\rceil$ we define the mapping:

$$f(v_{ij}) = \left\{ \begin{array}{ll} (j-1)m+i & \text{if } 1 \le j \le k \\ \\ k(m-1)+n(i-1)+j & \text{if } j=k+1,...,n \end{array} \right\}$$

It follows that:

$$e_f(1) = k(m-1) + m(n-k-1)$$

= $mn - (m+k)$

and

$$e_f(0) = 2mn - (m+n) - mn + (m+k)$$

= $mn - (n-k)$

 \mathbf{SO}

$$|e_f(0) - e_f(1)| = |mn - n + k - mn + m + k|$$
$$= |-n + 2k + m|$$
$$= \left\{ \begin{array}{ccc} 0 & if \ n - mis \ even\\ 1 & if \ n - m \ is \ odd \end{array} \right\}$$

Similarly if m > n we apply the same mapping but replacing i by j and m by n, i.e.:

$$k = \left\lceil \frac{1}{2}(m-n) \right\rceil \text{ and:}$$

$$f(v_{ij}) = \left\{ \begin{array}{ll} (i-1)n+j & \text{if } 1 \le i \le k \\ \\ k(n-1)+m(j-1)+i & \text{if } i=k+1,...,m \end{array} \right\}$$

Hence the grid graph $P_m \times P_n$ is a difference cordial graph for all m, n.

Example 2.22. Let $P_m \times P_n = P_4 \times P_3$

$$n = 3, m = 4, |V| = 12, |E| = 17$$

 $f(v_{ij}) = 3(i - 1) + j$
 $e_f(1) = 8, e_f(0) = 9.$



Figure 2.12: A difference cordial labeling for grid graph $P_4 \times P_3$

Example 2.23. Let $P_m \times P_n = P_5 \times P_8$

n=8, m=5, |V|=40, |E|=67, k=2, then



Figure 2.13: A difference cordial labeling for grid graph $P_5 \times P_8$

 $e_f(0) = 34, \ e_f(1) = 33$

2.3.4 Step ladder graph $S(T_n)$:

Definition 4. Let P_n be a path on n vertices denoted by (1,1), (1,2), ..., (1,n) and n-1 edges denoted by $e_1, e_2, ..., e_{n-1}$ where e_i is the edge joining the vertices (1,i) and (1,i+1). On each edge e_i , i = 1, 2, ..., n-1 we erect a ladder with n-(i-1) steps including the edge e_i . The graph obtained is called a step ladder graph and is denoted by $S(T_n)$, where n denotes the number of vertices in the base.

The Figure 4.2 shows the step ladder graph:



Figure 2.14: The step ladder graph $S(T_n)$.

The number of vertices and edges in the step ladder graph $S(T_n)$ are:

$$|V| = 2 + 3 + 4 + \dots + n + n$$
$$= \frac{1}{2}n(n+1) + (n-1)$$
$$= \frac{n^2 + 3n - 2}{2}$$

$$|E| = 2(|V| - n)$$

= $n(n + 1) - 2$

We notice for all step ladder graphs that $i + j \le n + 2$

Proposition 2.24. Every step ladder graph $S(T_n)$ is a difference cordial graph for all n.

Proof. Let $S(T_n)$ be a step ladder graph then

$$|E| = n(n+1) - 2 = n^2 + n - 2$$

Define the function:

 $f: V(S(T_n)) \longrightarrow \{1, 2, ..., \frac{1}{2}n(n+1) + (n-1)\}$ by:

$$f(v_{ij}) = \left\{ \begin{array}{ll} j + (i-1)n & 1 \le i \le 3\\\\ j + (i-1)n - \frac{1}{2}(i-3)(i-2) & i \ge 4 \end{array} \right\}$$

$$e_f(1) = (3n - 4) + (n - 3) + (n - 4) + (n - 5) + \dots + 3 + 2 + 1$$

= (n - 1) + (n - 1) + (n - 2) + (n - 3) + \dots + 2 + 1
= (n - 1) + \frac{1}{2}n(n - 1)
= \frac{1}{2}(n^2 + n - 2),

then $e_f(1) = \frac{1}{2} |E|$ which means $|e_f(1) - e_f(0)| = 0$

CHAPTER 2. ON DIFFERENCE CORDIAL GRAPHS

Therefor $S(T_n)$ is a difference cordial graph for all n

Example 2.25. In the following figure the difference cordial labeling for $S(T_{12})$ graph.



Figure 2.15: Difference cordial labeling for the step ladder graph $S(T_{12})$

2.3.5 Double Sided Step Ladder Graph $2S(T_{2n})$:

Definition 5. Let P_{2n} be a path of length 2n - 1 with 2n vertices (1,1), (1,2), ..., (1,2n) with (2n - 1) edges, $e_1, e_2, ..., e_{2n-1}$, where e_i is the edge joining the vertices (1,i) and (1,i + 1). On each edge e_i , for i = 1, 2, ..., n, we erect a ladder with (i + 1) steps including the edge e_i and on each edge e_i , for i = n + 1, n + 2, ..., 2n - 1, we erect a ladder with 2n + 1 - i steps including the edge e_i .

The double sided step ladder graph $2S(T_{2n})$ has vertices denoted by: (1,1), (1,2), ..., (1,2n), (2,1), (2,2), ..., (2,2n), (3,2), (3,3), ..., (3,2n-1), (4,3), (4,4), ..., (4,2n-2), ..., (n+1,n), (n+1,n+1). In the ordered pair (i, j), i denotes the row number (counted from bottom to top) and j denotes the column number (from left to right) in which the vertex occurs.

The figure 2.16 show the $2S(T_{10})$



Figure 2.16: Double sided step ladder graph $2S(T_{10})$

Proposition 2.26. The double sided step ladder graph $2S(T_m)$ is a difference cordial graph for all m, where m = 2n denotes the number of vertices in the base.

Proof. Let G = (V, E) be the double sided step ladder graph $2S(T_m)$ where m = 2n, then

$$|V| = n^2 + 3n$$
 and $|E| = 2n^2 + 3n - 1$

Define $f: V \longrightarrow 1, 2, ..., n^2 + 3n$ by:

$$f(v_{i,j}) = \begin{cases} u & if \quad i = 1 \text{ and } j \leq \left\lceil \frac{1}{2}n \right\rceil \\ j + 2n(i-1) & if \quad i = 1 \text{ and } j \geq \left\lceil \frac{1}{2}n \right\rceil + 1 \\ j + 2n(i-1) & if \quad i = 2 \\ j + 2n(i-1) - (i-1)^2 & if \quad i = 3, 4, \dots, n+1 \end{cases}$$

where

$$u = \begin{cases} 2j(\operatorname{mod}\left\lceil\frac{1}{2}n\right\rceil + 1) & \text{if } n \equiv 3 \text{ or } n \equiv 0 \pmod{4} \\\\ 2j(\operatorname{mod}\left\lceil\frac{1}{2}n\right\rceil + 1) + \left\lfloor\frac{2j}{\left\lceil\frac{1}{2}n\right\rceil + 1}\right\rfloor & \text{if } n \equiv 1, 2(\operatorname{mod} 4) \\\\ (2j-1)(\operatorname{mod}\left\lceil\frac{1}{2}n\right\rceil + 1) + 2\left\lfloor\frac{2j}{\left\lceil\frac{1}{2}n\right\rceil + 1}\right\rfloor & \text{if } n \equiv 3(\operatorname{mod} 4) \end{cases}$$

from the last three parts of the definition of f we will get $n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1$ edges give are labeled 1, while in the first part all edges are labeled 0 except when $\left\lceil \frac{1}{2}n \right\rceil \leq 4$ we will get an edge is labeled 1 since $1 \leq u \leq 3$

Case 1:

If
$$n = 2$$
 then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 7$ and $e_f(0) = 6$,
if $n = 3$ then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 13$ and $e_f(0) = 13$,

if n = 4 then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 22$ and $e_f(0) = 21$ and if n = 5 then $e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1 = 32$ and $e_f(0) = 32$. Case 2:

If $n \equiv 0 \pmod{4}$ then n = 4k for some positive integer number k and $\left\lceil \frac{1}{2}n \right\rceil = 2k$, then $|E| = 2(4k)^2 + 3(4k) - 1 = 32k^2 + 12k - 1$ and

$$\begin{split} f(v_{1\left\lceil \frac{1}{2}n\right\rceil}) &= 2\left\lceil \frac{1}{2}n \right\rceil \pmod{\left(\left\lceil \frac{1}{2}n \right\rceil + 1\right)} \\ &= (2*2k) \pmod{(2k+1)} = 4k \pmod{(2k+1)} \\ &= 2k-1 \end{split}$$

while $f(v_{1\lceil \frac{1}{2}n\rceil+1}) = \lceil \frac{1}{2}n\rceil + 1 = 2k + 1$, thus the label of the edge $v_{1\lceil \frac{1}{2}n\rceil}v_{1\lceil \frac{1}{2}n\rceil+1}$ will be included in $e_f(0)$, therefor $e_f(1) = n^2 + 2n - \lceil \frac{1}{2}n\rceil - 1 = 16k^2 + 8k - 2k - 1 = 16k^2 + 6k - 1$ and

$$e_f(0) = |E| - e_f(1) = 32k^2 + 12k - 1 - 16k^2 - 6k + 1 = 16k^2 + 6k$$

we get $|e_f(0) - e_f(1)| = 1$

Case 3:

If $n \equiv 1 \pmod{4}$ then, n = 4k + 1 for some positive integer number kand $\left\lceil \frac{1}{2}n \right\rceil = 2k + 1$ then $|E| = 2(4k + 1)^2 + 3(4k + 1) - 1 = 32k^2 + 28k + 4$, and

$$\begin{aligned} f(v_1 \lfloor \frac{1}{2}n \rfloor) &= 2\left(\left\lceil \frac{1}{2}n \right\rceil \right) \left(\operatorname{mod}\left(\left\lceil \frac{1}{2}n \right\rceil + 1 \right) \right) + \left\lfloor \frac{2 \lfloor \frac{1}{2}n \rceil}{\lfloor \frac{1}{2}n \rceil + 1} \right\rfloor \\ &= 2(2k+1)(\operatorname{mod}(2k+2)) + \left\lfloor \frac{2 \lfloor 2k+1 \rceil}{\lfloor 2k+1 \rceil + 1} \right\rfloor \\ &= (4k+2) \left(\operatorname{mod}(2k+2) \right) + 1 \\ &= 2k+1 \end{aligned}$$

while $f(v_1 \lfloor \frac{1}{2}n \rfloor + 1) = \lfloor \frac{1}{2}n \rfloor + 1 = 2k + 2$, thus the label of the edge $v_1 \lfloor \frac{1}{2}n \rfloor v_1 \lfloor \frac{1}{2}n \rfloor + 1$ will be included in $e_f(1)$, therefor $e_f(1) = n^2 + 2n - \lfloor \frac{1}{2}n \rfloor - 1 + 1 = (4k+1)^2 + 2(4k+1) - 2k - 1 - 1 + 1 = 16k^2 + 14k + 2$ and $e_f(0) = |E| - e_f(1) = 32k^2 + 28k + 4 - 16k^2 - 14k - 2 = 16k^2 + 14k + 2$

we get $|e_f(0) - e_f(1)| = 0$

Case 4:

If $n \equiv 2 \pmod{4}$ then n = 4k + 2 for some positive integer number kand $\left\lceil \frac{1}{2}n \right\rceil = 2k + 1$ then $|E| = 2(4k+2)^2 + 3(4k+2) - 1 = 32k^2 + 44k + 13$ and

$$\begin{split} f(v_{1\left\lceil\frac{1}{2}n\right\rceil}) &= (2\left\lceil\frac{1}{2}n\right\rceil)(\mathrm{mod}\left\lceil\frac{1}{2}n\right\rceil + 1) + \left\lfloor\frac{2\left\lceil\frac{1}{2}n\right\rceil}{\left\lceil\frac{1}{2}n\right\rceil + 1}\right\rfloor \\ &= 2(2k+1)(\mathrm{mod}(2k+2) + \left\lfloor\frac{2\left\lceil2k+1\right\rceil}{\left\lceil2k+1\right\rceil + 1}\right\rfloor \\ &= (4k+2)(\mathrm{mod}(2k+2)) + 1 \\ &= 2k+1 \end{split}$$

while $f(v_1 \lfloor \frac{1}{2}n \rfloor + 1) = \lfloor \frac{1}{2}n \rfloor + 1 = 2k + 2$, thus the label of the edge $v_1 \lfloor \frac{1}{2}n \rceil v_1 \lfloor \frac{1}{2}n \rceil + 1$ will included in $e_f(1)$, therefore

$$e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1$$

= $(4k+2)^2 + 2(4k+2) - 2k - 1 - 1 + 1$
= $16k^2 + 22k + 7$

and

$$e_f(0) = |E| - e_f(1)$$

= $32k^2 + 44k + 13 - 16k^2 - 22k - 7$
= $16k^2 + 22k + 6$

we get $|e_f(0) - e_f(1)| = 1$

Case 5:

If $n \equiv 3 \pmod{4}$, n = 4k + 3 for some positive integer number k and $\left\lceil \frac{1}{2}n \right\rceil = 2k + 2$, then

$$|E| = 2(4k+3)^2 + 3(4k+3) - 1 = 32k^2 + 60k + 26$$

and

$$\begin{split} f(v_1 \lfloor \frac{1}{2}n \rceil) &= (2 \lfloor \frac{1}{2}n \rceil - 1) \pmod{\left(\lfloor \frac{1}{2}n \rceil + 1 \right)} + 2 \lfloor \frac{2 \lfloor \frac{1}{2}n \rceil - 1}{\lfloor \frac{1}{2}n \rceil + 1} \\ &= (2(2k+2) - 1) \pmod{(2k+3)} + 2 \lfloor \frac{2 \lfloor 2k+1 \rceil - 1}{\lfloor 2k+1 \rceil + 1} \rfloor \\ &= (4k+3) \pmod{(2k+3)} + 2 \\ &= 2k+2, \end{split}$$

while $f(v_1 \lfloor \frac{1}{2}n \rfloor + 1) = \lfloor \frac{1}{2}n \rfloor + 1 = 2k + 3$, thus the label of the edge $v_1 \lfloor \frac{1}{2}n \rceil v_1 \lfloor \frac{1}{2}n \rceil + 1$ will be included in $e_f(1)$, therefor

$$e_f(1) = n^2 + 2n - \left\lceil \frac{1}{2}n \right\rceil - 1 + 1$$

= $(4k+3)^2 + 2(4k+3) - 2k - 2 - 1 + 1$
= $16k^2 + 30k + 13$

and

$$e_f(0) = |E| - e_f(1)$$

= $32k^2 + 60k + 26 - 16k^2 - 30k - 13$
= $16k^2 + 30k + 13$

we get $|e_f(0) - e_f(1)| = 0.$

From the cases 1, 2, 3, 4 and 5 we conclude that the double sided step ladder graph $2S(T_{2n})$ is a difference cordial graph for all integer number n

We discuss here some types of graphs not always difference cordial such as diagonal ladder graph, diagonal grid graph and friendship graph.

Diagonal ladder graph is a ladder with additional edges $u_i v_{i+1}$ and $u_{i+1}v_i$, denoted by DL_n , where n is half its vertices and the number of its edges is 5n - 4

Corollary 2.27. The diagonal ladder graphs are difference cordial if $n \leq 3$.

Proof. Let the graph G be the diagonal ladder graph DL_n with 2n vertices that means there are 5n - 4 edges in G, G is a difference cordial graph. Then by proposition 2.6 we get,

$$5n - 4 \le 2(2n) - 1$$
$$n \le 3$$

then the diagonal ladder graph is difference cordial when n = 2 or n = 3

Example 2.28 shows that DL_2 and DL_3 are difference cordial.

Example 2.28. The following are labeling for the diagonal ladder graphs DL_2, DL_3



Figure 2.17: The difference cordial labelings for the diagonal ladder graphs DL_2 and DL_3 .

The graph $P_m \times P_n$ with diagonal edges is called diagonal grid graph and denoted by $D(P_m \times P_n)$. It has mn vertices and 2(2mn+1) - 3(m+n) edges.

Remark 2.29. Diagonal grid graph $P_m \times P_n$ are not difference cordial graphs for both $m, n \ge 3$

Proof. Let $G = D(P_m \times P_n)$, from Proposition 2.1 if G is a difference cordial graph then $q \leq 2p - 1$. Let m = n = 3, then

$$q = 2(2mn + 1) - 3(m + n)$$

= 2(2 \cdot 3 \cdot 3 + 1) - 3(3 + 3)
= 20 \ne 17

then $D(P_m \times P_n)$ cannot be a difference cordial graph for both $m, n \geq 3$.

This is consistent with corollary 2.27 since diagonal ladder graphs are diagonal grid graphs.

Another type of graphs will be discussed here named one-point union fan graph, where a graph G in which a vertex distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph, the Graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called a one-point union of the n copies of G.

Proposition 2.30. The fan graph F_n is difference cordial for all n.[23]

Proposition 2.31. The one-point union $F_n^{(m)}$ of m copies of a fan F_n is difference cordial for all n and for $m \leq 5$.

Proof. Let $G = F_n^{(m)}$, then |V(G)| = mn + 1 and |E(G)| = m(2n - 1). These vertices are : the central vertex is denoted by v_{00} and the other vertices are denoted by v_{ij} , $1 \le i \le n$ and $1 \le j \le m$, as in the Figure 2.18.



Figure 2.18: The graph $F_n^{(m)}$.

For each copy of a fan F_n there are n-1 edges labeled 1, therefore there are m(n-1) + 2, edges are labeled 1 in $F_n^{(m)}$ where the central vertex is labeled 1 (mod n) but is neither 1 nor mn + 1 then

$$e_f(0) = m(2n-1) - m(n-1) - 2 = mn - 2$$

Now

$$|e_f(0) - e_f(1)|$$

= $|mn - 2 - m(n - 1) - 2|$
= $|m - 4|$

then $|e_f(0) - e_f(1)| \ge 2$ for all $m \ge 6$.

We define the mapping f for $m\leq 5$ and $n\in\mathbb{N}$ by

$$f(v_{0\ 0}) \equiv 1 \pmod{n}$$
 and $f(v_{0\ 0}) \neq 1, mn+1$

and

$$f(v_{i \ j}) = \left\{ \begin{array}{ccc} (j-1)n+i & if \quad (j-1)n+i < f(v_{0 \ 0}) \\ \\ (j-1)n+i+1 & if \quad (j-1)n+i > f(v_{0 \ 0}) \end{array} \right\}$$

for all $i, j; 1 \le i \le n, 1 \le j \le m$.

As a special case, the friendship graph denotes by $F_2^{(m)}$ consists of one vertex union with m copies of paths P_2 consisting of 2m + 1 vertices and 3m edges as shown in Figure 2.19



Figure 2.19: The friendship graph F_5 .

Therefore the friendship graph $F_2^{(m)}$ is difference cordial iff $m \leq 5$.

Chapter 3

Some Results and Examples on Difference Cordial Graphs

In this chapter we introduce some results on difference cordial graphs and describe the difference cordial labeling for some families of graphs.

3.1 Introduction

Through this chapter we will deal with finite simple undirected graphs. By G = (V, E) we mean a finite undirected graph with p vertices and qedges where p = |V| and q = |E|. For standard terminology and notations we follow Harary [14], and for more details of labeling see [12]

R. Ponraj, S. Sathish Narayanan and R. Kala^[25], firstly, introduced the concept of difference cordial labeling in 2013. After that, they introduced many concepts and studied some types of graphs that have this kind of labeling such as: path, cycle, complete graph, complete

CHAPTER 3. SOME RESULTS AND EXAMPLES ON DIFFERENCE CORDIAL GRAPHS

bipartite graph, bistar, wheel, web, sunflower graph, lotus inside a circle, pyramid, permutation graph, book with n pentagonal pages, t-fold, wheel, double fan and some more standard graphs have been investigated in [23, 24, 25, 26, 27]. Within this area M. A.Seoud and Shakir M. Salman introduced some results and investigated some difference cordial graphs: ladder, step ladder,two sided step ladder,diagonal ladder,triangular ladder, grid graph and some types of one-point union graphs[34]

Definition 6. [14] The line graph L(G) of a graph G has a vertex for each edge of G, and two of these vertices are adjacent if and only if the corresponding edges in G have a common vertex.

Definition 7. [14] A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face; we usually choose this face to be exterior. An outerplanar graph G is maximal outerplanar if no line can be added without losing outerplanarity.

The maximal outerplanar graph is denoted by MOG in this thesis. The MOG has the following propositionerties:

Lemma 3.1. [14] Let G be an MOG with n vertices; $n \ge 3$, then:

- 1. there are 2n 3 edges, in which there are n 3 chords;
- 2. there are n-2 inner faces and each inner face is triangular;
- 3. there are at least two vertices with degree 2;
- 4. connectivity of G, k(G) is equal to 2.

Definition 8. [2] Let G = (V, E) be a graph and $u \in V$ a vertex of G. The open neighborhood of u or just the neighborhood of u, denoted by $N_G(u)$ or just N(u), is the set of all of the neighbors of u in G. Likewise, the closed neighborhood of u, denoted by $N_G[u]$ or just N[u], is the set of neighbors of u together with u itself.

Definition 9. [43] Duplication of a vertex v_i by a new edge $e = v'_i v''_i$ in a graph G produces a new graph G' such that $N(v'_i) \cap N(v''_i) = \{v_i\}$.

Definition 10. [9] A shell graph is defined as a cycle C_n with (n-3) chords sharing a common end point called the apex, shell graphs are denoted as C(n, n-3).

Definition 11. [42] A bow graph is defined to be a double shell in which each shell has any order.

Definition 12. [42] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.

Definition 13. [12] Define the shell-flower graph as k copies of the union of the shell C(n, n - 3) and K_2 where one end vertex of K_2 is joined to the apex of the shell. Figure 3.7 shows this type of graphs.

3.2 Some Results

Proposition 3.2. The graph G = (V, E) is difference cordial if and only if there exist some disjoint paths, such that their total length is more than or equal to $\left|\frac{1}{2}|E|\right|$.

Proof. Let G = (V, E) be a difference cordial graph with a mapping f, then $|e_f(1) - e_f(0)| \leq 1$, i.e., $e_f(1) - 1 \leq e_f(0) \leq e_f(1) + 1$ and $e_f(1) + e_f(0) = |E|$.

If |E| is even, then $e_f(1) = e_f(0)$ and $e_f(1) = \frac{1}{2} |E|$, this means $\frac{1}{2} |E|$ edges join the vertices labeled i, i + 1. Then there are some paths such that, the sum of their lengths is $\frac{1}{2} |E|$ which is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$.

If |E| is odd, then $e_f(1) = e_f(0) + 1$ or $e_f(1) = e_f(0) - 1$, if $e_f(1) = e_f(0) + 1$, then $e_f(1) = \lfloor \frac{1}{2} |E| \rfloor$, thus there are $\lfloor \frac{1}{2} |E| \rfloor$ edges join the vertices labeled i, i+1; in other words, there are some paths such that, the sum of their lengths is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$.

If $e_f(1) = e_f(0) - 1$, then $e_f(1) = \lfloor \frac{1}{2} |E| \rfloor$, thus there are $\lfloor \frac{1}{2} |E| \rfloor$ edges join the vertices labeled i, i + 1, this means the existence of some disjoint paths, as above, the sum of their lengths is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$.

Thus, if G = (V, E) is a difference cordial graph, then there exist some disjoint paths, the sum of their lengths is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$. It is necessary that the paths are disjoint otherwise there are three paths having a common vertex, which is impossible, since if this common vertex has the label x, two of the adjacent vertices should have the labels x - 1, x + 1. But the third vertex can't take either x - 1 or x + 1, but something else.

Suppose there exist some disjoint paths, the sum of their lengths is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$ on the graph G = (V, E). If there is only one such path and we labeled its vertices by $i, i + 1, ..., \lfloor \frac{1}{2} |E| \rfloor$, then all edges of this path are labeled 1. If there are two disjoint paths their lengths are k and h where k + h is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$, then we label these paths by j, j+1, ..., j+kand t, t+1, ..., t+h, respectively, hence there are k+h edges labeled 1, continue this procedure with all paths.

Then G = (V, E) is difference cordial.

Proposition 3.3. If the graph G = (V, E) is semi-Hamiltonian, then G is a difference cordial graph if and only if the length of the semi-Hamiltonian path is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$.

Proof. Let G be a (p,q) graph containing a semi-Hamiltonian path where q = |E|, and let G be a difference cordial graph, since G contains a semi-Hamiltonian path, we can label it such that $e_f(1) = p - 1$ and according to Proposition 2.1;

$$q \le 2p - 1 \Longrightarrow p - 1 > \left\lfloor \frac{1}{2}q \right\rfloor - 1$$

G is difference cordial implies the length of the semi-Hamiltonian path is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$.

Suppose G contains a semi-Hamiltonian path whose length is more than or equal to $\lfloor \frac{1}{2} |E| \rfloor$, then G is difference cordial by Proposition 3.2.

Proposition 3.4. Every connected graph G(p,q) with q = 2p - 1 is difference cordial if and only if G is semi-Hamiltonian.

Proof. Let G(p,q) be undirected simple connected with q = 2p - 1.

Suppose G is a difference cordial graph then there exists labeling f such that $|e_f(1) - e_f(0)| \leq 1$. From Proposition 3.2 there is a path with length at least p-1, but by [42] the maximum length is p-1, then this path must pass through all vertices of graph G, i.e., G is semi-Hamiltonian.

If G is a semi-Hamiltonian graph, we label the vertices on a semi-Hamiltonian path by a sequence of integers 1, 2, ..., p we will get p-1 edges labeled 1 and the other edges are labeled 0, this means G is difference cordial.

Corollary 3.5. The Peterson graph is difference cordial.

Proof. Direct consequence of Proposition 3.3.

Proposition 3.6. Every biconnected outerplanar graph is a difference cordial graph.

Proof. Let G(p,q) be a biconnected outerplanar graph, then G is a Hamiltonian graph thus there is a semi-Hamiltonian path in G.

- **Case 1:** If G is a maximal outerplanar graph, then q = 2p 3 thus $q \leq 2p 1$ and we label the vertices in this path by a sequence of integer numbers so we get p 1 of edges labeled 1, and other edges will be labeled 0, thus $e_f(0) = p 2$, means G is difference cordial.
- **Case 2:** If G is not a maximal outerplanar graph, then $q \leq 2p 3$; thus $q \leq 2p 1$, then the semi-Hamiltonian path is of length p 1,

which is more than $\lfloor \frac{1}{2}q \rfloor$, then by Proposition 3.3, G is difference cordial.

Then from Case 1 and Case 2 every biconnected outerplanar graph is difference cordial. $\hfill \Box$

Example 3.7. The following outerplanar graphs with their difference cordial labeling are shown in Figure 3.1 and Figure 3.2 respectively,

p = 12, q = 21 and $e_f(0) = 10$, $e_f(1) = 11$



Figure 3.1: A difference cordial labeling for the maximal outerplanar graph with 12 vertices.



Figure 3.2: An outerplanar graph with 12 vertices and 20 edges.

p = 12, q = 20 and $e_f(0) = 10$, $e_f(1) = 10$.

- **Proposition 3.8.** 1. The line graph L(G) for any graph G with $\delta(G) \geq 3$ cannot be difference cordial.
 - 2. The line graph L(G) for any graph G with $d(G) \ge 3$ cannot be difference cordial.
- *Proof.* 1. Let G(p,q) be a simple undirected graph with $\delta(G) \ge 3$, and L(G) its line graph.

The number of vertices on L(G) is equal to q, each of these edges in L(G) has on its ends two vertices with degree more than or equal to 3, then the vertex corresponding with this edge in the line graph L(G) will be of degree more than 4, then L(G) is a graph with $\delta(G) \geq 4$ then by Proposition 2.3, it cannot be difference cordial.

2. It follows directly from 1.

Remark 3.9. The union of two disjoint difference cordial graphs need not be difference cordial.

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two disjoint difference cordial graphs, where f_1 and f_2 are their labelings, then $|e_{f_1}(0) - e_{f_1}(1)| \leq 1$ and $|e_{f_2}(0) - e_{f_2}(1)| \leq 1$.

Since if $e_{f_1}(0) = e_{f_1}(1) + 1$, and $e_{f_2}(0) = e_{f_2}(1) + 1$, and G_1 , G_1 are disjoint graphs, then $G_1 \cup G_2$ has $q = q_1 + q_2$ and $p = p_1 + p_2$, then $e_{f_1}(0) + e_{f_2}(0) = e_{f_1}(1) + e_{f_2}(1) + 2$, hence $G_1 \cup G_2$ is not difference cordial.

Example 3.10. The following two disjoint difference cordial graphs are shown in Figure 3.3.



Figure 3.3: Two disjoint difference cordial graphs.

3.3 Difference Cordial Labeling for Some Families of Graphs

In this section we introduce difference cordial labeling for some types of graphs.

3.3.1 Graph Obtained by Duplication of Vertex by an Edge

Here we discuss only the graph obtained by duplication of each vertex of C_n by an edge.

Proposition 3.11. The graph obtained by duplication of each vertex of C_n by an edge is difference cordial.
Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n and G be the graph obtained by duplication of each vertex v_i of the cycle C_n by an edge $u_i u_{i+1}$ $(1 \le i \le n)$.

Then $V(G) = V(C_n) \cup \{u_1, u_2, ..., u_{2n}\}$ and $E(G) = E(C_n) \cup \{u_{2i-1}v_i, u_{2i}v_i, u_{2i}u_{2i-1}; 1 \le i \le n\}$, then V(G) = 3n and E(G) = 4n.

Define the mapping $f: V(G) \longrightarrow \{1, 2, ..., 3n\}$ by:

$$f(u_i) = i \quad \& \quad f(v_i) = \left\{ \begin{array}{cc} i + 1 + 2n & , \ i \neq n \\ 2n + 1 & , \ i = n \end{array} \right\}$$

From the definition of $f(u_i)$ there are n edges labeled 1 and from $f(v_i)$ there are n-1 edges labeled 1, also the edge $u_{2n}v_n$ is labeled 1, then $e_f(1) = 2n$ and G is a difference cordial graph.

Example 3.12. The graph obtained by duplication of vertex of C_7 by an edge with its difference cordial labeling is shown in Figure 3.4:



Figure 3.4: The difference cordial labeling for the graph obtained by duplication of vertex of C_7 by an edge.

3.3.2 Bow Graphs

The bow graph G(p,q) could be described as follows: In graph G, the shell that is present to the left of the apex is called the left wing and the shell that is present to the right of the apex and it is considered as the right wing.

Figure 3.5 shows the bow graph with shells of orders m and n excluding the apex.

Proposition 3.13. All bow graphs are difference cordial.

Proof. Let G be a bow graph with two shells of orders m and n excluding the apex. Then, the number of vertices in G is p = m + n + 1 and the number of edges q = 2(m + n - 1). The apex of the bow graph is denoted by v_0 , denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$.

Define the mapping of labeling

 $f: V \longrightarrow \{1, 2, ..., m + n + 1\}$ by:

$$f(v_i) = \left\{ \begin{array}{cc} i & , i \neq 0 \\ m+n+1 & , i = 0 \end{array} \right\}$$

From the above definition we see there are m-1 edges labeled 1 and m edges labeled 0 in the right wing of the bow graph, also there are n edges labeled 1 and n-1 edges labeled 0 in the left wing of the bow graph. Then $|e_f(1)| = |e_f(0)| = m+n-1$, which implies the bow graph is difference cordial.

Example 3.14. The bow graph G with two wings having m, n vertices respectively, with its difference cordial labeling is shown in Figure 3.5.



Figure 3.5: The bow graph with m + n + 1 vertices and its difference cordial labeling.

3.3.3 Butterfly Graphs

Proposition 3.15. The butterfly graphs are difference cordial.

Proof. Let G be a butterfly graph with shells of orders m and n excluding the apex, then the number of vertices in G is p = m+n+3 and the number of edges q = 2(m+n). The apex of the butterfly graph is denoted as v_0 , denote the vertices in the right wing of the butterfly graph from bottom to top by $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the pendant vertices in the pendant edges are denoted by v_{m+n+1}, v_{m+n+2} .

Define the mapping of labeling $f:V\longrightarrow \{1,2,...,m+n+3\}$ by:

$$f(v_i) = \begin{cases} i & , 1 \le i \le n+m \\ m+n+1 & , i=0 \\ i+1 & , n+m < i \le n+m+2 \end{cases}$$

From the above definition we see there are m-1 edges labeled 1 and m edges labeled 0 in the right wing of the bow graph, also there are n edges labeled 1 and n-1 edges labeled 0 in the left wing of the butterfly graph, while the pendant edges are labeled 1 and 0.

Then $|e_f(1)| = |e_f(0)| = m + n$, which implies the butterfly graph is difference cordial.

Example 3.16. The butterfly graph G with two wings having m, n vertices respectively, and its difference cordial labeling is shown in Figure 3.6.



Figure 3.6: A difference cordial labeling for the butterfly graphs.

3.3.4 Shell-Flower Graphs

From the definition of the shell-flower graph it contains k copies of the shell C(n, n - 3) and k copies of K_2 where one vertex of K_2 is joined to the apex of the shell and each shell in the shell-flower graph is called a petal, hence it consists of k petals and k pendant edges.

Proposition 3.17. The shell-flower graph cannot be difference cordial when $k \geq 3$ for all n.

Proof. Let G(p,q) be the shell-flower graph with k petals in each one C(n, n-3), and then p = nk + 1, and q = 2k(n-1) as in Figure 3.7.



Figure 3.7: A shell-flower graph with k petals.

The apex vertex has degree p - 1, then for any labeling of vertices there are at least p - 3 = nk - 2 edges labeled 0 but the number of the total other edges is:

$$2k(n-1) - nk + 2$$

= (nk - 2) + (4 - 2k),

assume all these edges are labeled as 1. Then for all $k \geq 3$, $e_f(0) \geq e_f(1) + 2$. Hence the shell-flower graph is not difference cordial for all $k \geq 3$.

Example 3.18. The shell-flower graph with two petals C(9,6) is shown in Figure 3.8.



Figure 3.8: A difference cordial labeling for the shell-flower graph with two petals.

3.3.5 One-Point Union of Complete Graphs

In this subsection we discuss the difference cordial labeling of the onepoint union of m complete graphs K_n of order n, as in Figure 3.9.



Figure 3.9: The graph $K_5^{(2)}$.

Proposition 3.19. Let $K_n^{(m)}$ be the one-point union of m complete graphs K_n .

- 1. $K_2^{(m)}$ is difference cordial when $m \leq 5$.
- 2. $K_3^{(m)}$ is difference cordial when $m \leq 5$.
- 3. $K_4^{(m)}$ is difference cordial when $m \leq 2$.
- 4. $K_n^{(m)}$ is not difference cordial for all $n \ge 5$.
- *Proof.* 1. $K_2^{(m)}$ is star graph from [37] it is difference cordial when $m \leq 5$.
 - 2. $K_3^{(m)}$ is friendship graph from [37] it is difference cordial when $m \leq 5$.
 - 3. Let G = K₄^(m), then G has 3m + 1 vertices and 6m edges, so there is one vertex say v₀ adjacent with all other vertices in G, so the graph G-v₀ consists of m components each component is a triangle which consists of at most 2 edges labeled 1 and at least one edge labeled 0, i.e., there are at most 2m edges labeled 1 and at least m edges labeled 0. But v₀ is adjacent with all other vertices, hence its degree is 3m, then G contains at least 4m − 2 edges labeled 0 and at most 2m + 2 edges labeled 1 and we have:

$$e_f(0) - e_f(1) \ge (4m - 2) - (2m + 2)$$

 $\ge 2m - 4$

If $m \ge 3$ then $e_f(0) - e_f(1) \ge 2$, thus $K_4^{(m)}$ is not difference cordial for all $m \ge 3$.

4. In graph $K_n^{(m)}, \delta(K_n^{(m)}) \ge 4$ when $n \ge 5$, then by Proposition 2.3, $K_n^{(m)}$ is not difference cordial.

Chapter 4

Some Results on Divisor Cordial Graphs

In this chapter we introduce some results on divisor cordial graphs and describe the divisor cordial labeling for some families of graphs.

4.1 Introduction

In this chapter by a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [14]. Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [12]. The concept of cordial labeling was introduced by Cahit [8], in [46], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graphs are presented in [29, 18, 22, 19, 20, 45, 44, 46, 47]. The brief summaries of Definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [7].

Definition 14. [46] Let G = (V(G), E(G)) be a simple graph and $f: V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if f(u)|f(v) or f(v)|f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \le 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition 15. [14] The neighborhood of a vertex u is the set $N_u(G)$ consisting of all vertices v which are adjacent with u. The closed neighborhood is $N_u[G] = N_u(G) \bigcup \{u\}$.

Definition 16. [17] The Jelly fish graph J(m, n) is obtained from a $4 - cycle v_1, v_2, v_3, v_4$ by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4 .

Definition 17. [9] A shell graph is defined as a cycle C_n with (n-3) chords sharing a common end point called the apex, shell graphs are denoted as C(n, n-3).

Definition 18. [42] A bow graph is defined to be a double shell in which each shell has any order.

Definition 19. [9] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.

4.2 The Results

Proposition 4.1. For any simple graph G(p,q), the maximum value of $e_f(1)$ is $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$, where $p \ge 4$.

Proof. Let G(p,q) be a simple connected graph and let the vertex v_k be of maximum degree $\Delta(G)$, if we labeled this vertex by 1 then we will achieve $\Delta(G)$ edges labeled 1, and from division algorithm the maximum number of the multiples of labels of vertices are:

for 2 is $\lfloor \frac{p}{2} \rfloor - 1$, for 3 is $\lfloor \frac{p}{3} \rfloor - 1$, for 4 is $\lfloor \frac{p}{4} \rfloor - 1$, . . for $\lfloor \frac{p}{2} \rfloor$ is $\lfloor \frac{p}{\lfloor \frac{p}{2} \rfloor} \rfloor - 1$ which must equal 1 hence the maximum value for $e_f(1)$ equals $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$ in any graph G(p,q).

Corollary 4.2. For each r-regular graph the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$; where $k = \lfloor \frac{p}{r+1} \rfloor$ and $p \ge 4$.

Proof. Let G(p,q) be an r-regular graph then $\triangle(G) = r$, and for each vertex v in graph G the maximum number of edges that label 1 in $N_v(G)$ is r, hence for all i in which $\lfloor \frac{p}{i} \rfloor - 1 \ge r$ we reduced it to r.

But from Proposition 4.1 the maximum value of $e_f(1)$ is

 $\triangle(G) + \sum_{i=2}^{\left\lfloor \frac{p}{2} \right\rfloor} (\lfloor \frac{p}{i} \rfloor - 1), \text{ then the maximum value in an } r - regular \text{ graph}$ is:

$$= r + \sum_{i=2}^{k} (r) + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$$
$$= r + (k-1)r + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$$
$$= kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$$

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Proof. Let G(p,q) be a divisor cordial graph, then $|e_f(0) - e_f(1)| \leq 1$, means $e_f(0) = e_f(1) - 1$ or $e_f(0) = e_f(1)$ or $e_f(0) = e_f(1) + 1$, by proposition 4.1,

$$q \leq 2e_f(1) + 1$$

$$q \leq 2(\Delta G + \sum_{i=2}^{\left\lfloor \frac{p}{2} \right\rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)) + 1$$

$$q \leq 2(\Delta G + \sum_{i=3}^{\left\lfloor \frac{p}{2} \right\rfloor} (\left\lfloor \frac{p}{i} \right\rfloor)) + 3$$

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4.3 Divisor Cordial Labeling for Some Families of Graphs

In this section we introduce the divisor cordial labeling for some types of graphs.

4.3.1 The Jelly Fish Graph

Proposition 4.4. For $m, n \ge 1$, Jelly fish graph J(m, n) is a divisor cordial graph.

Proof. Let G(V, E) = J(m, n). Then G has (m + n + 4) vertices and (m + n + 5) edges.

Without losing of generality, Let $m \leq n$. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}, V_2 = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = E_1 \cup E_2$, where $E_1 = \{xu, uy, yv, vx, xy\}, E_2 = \{uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define $f: V \to \{1, 2, ..., (m+n+4)\}$ as follows:

f(u) = 1, f(v) = 2, f(x) = m + n + 4, f(y) = m + n + 3 and $f(u_i) = 2(i + 1); i = 1, 2, ..., m$,

$$f(v_i) = \begin{cases} 2i+1 , i = 1, 2, ..., m \\ i+m+2 , i = m+1, m+2, ..., n \end{cases}$$

From the function f there are m + 2 edges labeled 1 sice f(u) = 1, and since f(V) = 2 there are exactly $\lfloor \frac{1}{2}(n-m) \rfloor$ of pendent edges from vlabeled 1 and only one from vx or vy. means $e_f(1) = m + 3 + \lfloor \frac{1}{2}(n-m) \rfloor$ and

$$e_f(0) = m + n + 5 - (m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor)$$

= $n + 2 - \lfloor \frac{1}{2}(n - m) \rfloor$,

Case 1: m, n are odd

The |E| and $\lfloor n - m \rfloor$ are even, hence, $|e_f(0) - e_f(1)| = 1$

Case 2: m, n are even

The |E| is odd and $\lfloor n - m \rfloor$ is even, hence, $|e_f(0) - e_f(1)| = 1$

Case 3: m is odd and n is even The |E| is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Case 4: m is odd and n is even

The |E| is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Then from Case 1, Case 2, Case 3 and Case 4 the jelly fish graph is divisor cordial. $\hfill \Box$

Example 4.5. The jelly fish graph j(6, 11) and its divisor cordial labeling are shown in Figure 4.1



Figure 4.1: A Jelly fish graph j(6, 11) and its divisor cordial labeling

4.3.2 The shell and The Bow Graph

Proposition 4.6. Every shell graph is divisor cordial.

Proof. Let G = (V, E) be a C(n, n - 3) graph with |V| = n, then |E| = 2n - 3 means |E| is an odd number, and let v_0 be the apex and $v_1, v_2, ..., v_{n-1}$ other its vertices.

Define the labeling $f:V\longrightarrow \{1,2,...,n\}$ as:

 $f(v_0) = 2, f(v_1) = 1$ and other vertices by the following:

	$2 \cdot 2$,	$2 \cdot 2^2$,	···· ,	$2 \cdot 2^{k_1}$,
3,	$3 \cdot 2$,	$3 \cdot 2^2$,	··· ,	$3 \cdot 2^{k_2}$,
5,	$5 \cdot 2$,	$5 \cdot 2^2$,	$\cdots,$	$5 \cdot 2^{k_3}$,
•		••••		···· ,
•		••••		··· ,

In this labeling, there are $\lceil \frac{n-1}{2} \rceil$ edges label 1 passing through v_0 , but other edges not passing through the apex make a path, hence there are also $\lfloor \frac{n-2}{2} \rfloor$ edges are labeled 1. Hence, $e_f(1) = \lceil \frac{n-1}{2} \rceil + \lfloor \frac{n-2}{2} \rfloor$

Case 1: *n* is odd, then $e_f(1) = \frac{n-1}{2} + \lfloor \frac{n-2}{2} \rfloor$ and $e_f(0) = \frac{n-1}{2} + \lceil \frac{n-2}{2} \rceil$

Case 2: n is even, then $e_f(1) = \lfloor \frac{n-1}{2} \rfloor + \frac{n-2}{2}$ and $e_f(0) = \lfloor \frac{n-1}{2} \rfloor + \frac{n-2}{2}$

In the two cases **Case 1** and **Case 2**, the difference between $e_f(1)$ and $e_f(0)$ is 1 which means the shell graph is divisor cordial.

Notice another divisor labeling for shell graph can found with fan graphs [46]

Example 4.7. The shell graph C(13, 10) and its divisor cordial labeling are shown in Figure 4.2



Figure 4.2: A shell graph C(13, 10) and its divisor cordial labeling

Proposition 4.8. All bow graphs are divisor cordial.

Proof. Let G be a bow graph with two shells of orders m and n excluding the apex. Then the number of vertices in G is p = m + n + 1 and the edges q = 2(m + n - 1). The apex of the bow graph is denoted by v_0 , denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$. Without lose of generality suppose $m \leq n$.

Define the labeling $f: V \longrightarrow \{1, 2, ..., m + n + 1\}$ by: $f(v_0) = 2, f(v_1) = 1$ and label the vertices of the wings by the following:

where $(2m-1) \cdot 2^{k_m} \leq n$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1) \cdot 2^a$ divides $(2m-1) \cdot 2^b (a < b)$ and $(2m-1) \cdot 2^{k_i}$ does not divide 2m+1.

Let G' be a graph obtained from the bow graph G by adding the edge $v_m v_{m+1}$.

The graph G' has an odd number of edges and it is a shell graph, then by Proposition 4.6 the graph G' is divisor cordial. The graph $G = G' - v_m v_{m+1}$ with even edges, then G is divisor cordial since:

- **Case 1:** If m + n is even, then $e_f(0) = e_f(1) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 0.
 - **Subcase i:** If $f(v_m) = (2t-1) \cdot 2^{k_i}$ for some *i*, then the deleted edge $v_m v_{m+1}$ is labeled 0.
 - **Subcase ii:** If $f(v_m) \neq (2t-1) \cdot 2^{k_i}$ for some *i*, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-l}$ in the wings, by *l* where *l* is the smallest integer satisfying $f(v_{m+1}) = (2t-1) \cdot 2^{k_i}$ for some *i*, and shift the labels of the vertices $v_{m+n-l+1}, v_{m+n-l+2}, ..., v_{m+n}$, by l+1 and take it modulo (m+n).

- **Case 2:** If m + n is odd, then $e_f(1) = e_f(0) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 1.
 - Subcase i: If $f(v_m) = (2t-1) \cdot 2^{k_i}$ for some *i*, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-1}$ in the wings, by one step and shift the label of vertex v_{m+n} by two and take it modulo (m+n).
 - Subcase ii: If $f(v_m) \neq (2t-1) \cdot 2^{k_i}$ for some *i*, then the edge $v_m v_{m+1}$ is labeled 1.

Then the bow graph G with two wings of m and n vertices is a divisor cordial graph for each m and n.

Example 4.9. The bow graph with two wings of 13 and 16 vertices respectively and its divisor cordial labeling are shown in Figure 4.3



Figure 4.3: A bow graph with m = 13, n = 16 and its divisor cordial labeling

4.3.3 Butterfly Graphs

Proposition 4.10. The butterfly graphs are divisor cordial.

Proof. Let G be a butterfly graph with shells of orders m and n excluding the apex then the number of vertices in G is p = m + n + 3 and the edges q = 2(m + n). The apex of the butterfly graph is denoted as v_0 , denote the vertices in the right wing of the butterfly graph from bottom to top as $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the vertices in the pendant edges are v_{m+n+1}, v_{m+n+2} .

Since the butterfly defined as a bow graph with exactly two pendent edges at the apex, then we define the labeling $f: V \longrightarrow \{1, 2, ..., m + n + 3\}$ by: $f(v_0) = 2, f(v_1) = 1, f(v_{m+n+1}) = m + n + 2, f(v_{m+n+2}) = m + n + 3$ and labeled the vertices of the wings by the following:

and we make the shift as in Proposition 4.8, for labeling of the vertices in the wings.

Since the only one of the numbers m + n + 2 or m + n + 3 must be even then the pendent edges will be labeled 1 and 0, hence the graph G is divisor cordial. **Example 4.11.** The butterfly graph G with two wings having m = 9, n = 15 vertices respectively, and its divisor cordial labeling is shown in Figure 4.4.



Figure 4.4: A divisor cordial labeling for the butterfly with 29 vertices

4.3.4 Friendship Graphs

The friendship graph F_n is a graph that can be constructed by coalescence *n* copies of the cycle graph C_3 of length 3 with a common vertex. The friendship graph F_n is isomorphic to the windmill graph Wd(3,n) [12]. The Friendship Theorem states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs [11]

Proposition 4.12. The friendship graphs are divisor cordial.

Proof. Let G(p,q) be a friendship graph F_n then the number of vertices in G is p = 2n + 1 and q = 3n edges.

Define $f: V \to \{1, 2, ..., (2n+1)\}$ as follows:

$$f(v_0) = 2, f(v_1) = 1, f(v_2) = 4, f(v_{2n}) = 2n + 1$$
 and

$$f(v_i) = \begin{cases} i & , i = 3, 5, ..., 2n - 1\\ 2(i - 1) & , i = 4, 6, 8, ..., 2(\lceil \frac{n}{2} \rceil + 1)\\ 4(\frac{i}{2} - \lfloor \frac{n}{2} \rfloor) & , i = 2(\lceil \frac{n}{2} \rceil + 1), 2(\lceil \frac{n}{2} \rceil + 1) + 2, ..., 2n - 2 \end{cases}$$

from definition of f there are n edges labeled 1 incident from the central vertex, also, $\lceil \frac{n}{2} \rceil$ edges are labeled 1 on other edges of the friendship F_n which are in the first $\lceil \frac{n}{2} \rceil$ triangles and all others edges are labeled 0, then $e_1 = n + \lceil \frac{n}{2} \rceil$ and $e_0 = n + \lfloor \frac{n}{2} \rfloor$, hence, the friendship F_n is divisor cordial for all n.

Example 4.13. The friendship graph F_6 and its divisor cordial labeling are shown in Figure 4.5



Figure 4.5: A friendship graph F_6 and its divisor cordial labeling.

Chapter 5

Divisor Cordial Labeling for Some Trees and Families of Graphs

In this chapter, we introduce divisor cordial labelings for some types of trees and some families of graphs.

5.1 Introduction

In this chapter by a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary[14]. Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [12]. The concept of cordial labeling was introduced by Cahit [8], in [46], Varatharajan et al. introduce the concept of divisor cordial labeling of graph.

The divisor cordial labeling of various types of graphs are presented in [29, 18, 22, 19, 20, 45, 44, 46, 47]. The brief summaries of definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [7].

Definition 20. [31]An m-star has a single root node with any number of paths of length m attached to it.

Definition 21. [3] A spider tree is a tree with at most one vertex of degree greater than 2. If such a vertex exists, it is called the branch point of the tree. A leg of a spider tree is any one of the paths from the branch points to a leaf of the tree.

Definition 22. [21] A k-distant tree consists of a main path called the spine, such that each vertex on the spine is joined by an edge to at most one path on k-vertices. Those paths are called tails. When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform k-distant tree.

Definition 23. [1] An olive tree has a root node with k branches attached: the i^{th} branch has length i.

Definition 24. [16] Let $G_1, G_2, ..., G_n$ be a family of disjoint stars. The tree obtained by joining a new vertex a to one pendant vertex of each star G_i is called a banana tree.

5.2 The Results

Proposition 5.1. All simple connected graphs with number of vertices less than eight are divisor cordial except K_4 .

Proof. In the following Figures: Figure 5.1, Figure 5.2 and Figure 5.3 We Shown all nonisomorphic graphs of 4, 5 and 6 with their divisor cordial graphs except K_4 and in Appendix A we will show the graphs with 7 vertices.



Figure 5.1: A divisor cordial labeling for all connected graphs with four vertices except K_4



Figure 5.2: A divisor cordial labeling for all connected graphs with five vertices.

CHAPTER 5. DIVISOR CORDIAL LABELING FOR SOME TREES AND FAMILIES OF GRAPHS

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Figure 5.3: A divisor cordial labeling for all connected graphs with six vertices.

CHAPTER 5. DIVISOR CORDIAL LABELING FOR SOME TREES AND FAMILIES OF GRAPHS

Proposition 5.2. Every r-regular graph with r > 6 is not divisor cordial graph.

Proof. Let G(p,q) be an r-regular graph. The minimum number of edges in r-regular graph with r > 6 is 28 in the complete graph K_8 , but from the Proposition 4.1 the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1);$ where $k = \lfloor \frac{p}{r+1} \rfloor$, that mean the maximum value of $e_f(1)$ in K_8 is:

$$kr + \sum_{i=k+1}^{\left\lfloor \frac{p}{2} \right\rfloor} \left(\left\lfloor \frac{p}{i} \right\rfloor - 1 \right), \quad k = \left\lfloor \frac{8}{8} \right\rfloor = 1$$
$$= 7 + \sum_{i=2}^{\left\lfloor \frac{8}{2} \right\rfloor} \left(\left\lfloor \frac{8}{i} \right\rfloor - 1 \right)$$
$$= 7 + \left(\left\lfloor \frac{8}{2} \right\rfloor - 1 \right) + \left(\left\lfloor \frac{8}{3} \right\rfloor - 1 \right) + \left(\left\lfloor \frac{8}{4} \right\rfloor - 1 \right)$$
$$= 12$$

We notice that: $\frac{1}{2} \cdot (28) \nleq 12 + 1$, which means the graph cannot be divisor cordial. Thus all r- regular graphs with r more than 6 are not divisor cordial.

5.3 Divisor Cordial Labeling for Some Trees

In this section we introduce the divisor cordial labeling for some types of trees such as olive, spider, m - stars, banana and caterpillar tree.

5.3.1 Olive Tree

Proposition 5.3. All olive trees are divisor cordial graphs.

Proof. Let ol_n be an olive tree of n branches, then the numbers of its edges and vertices are $|E| = \frac{n(n+1)}{2}$ and |V| = |E| + 1, respectively, denotes the vertices as: $v_0, v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{3,3}, \dots, v_{n,1}, v_{n,2}, \dots, v_{n,n}$.



Figure 5.4: The olive tree ol_n

Define the labeling of vertices as: $f(v_0) = 2, f(v_{1,1}) = 1$ and label the vertices $v_{n,1}, v_{n,2}, ..., v_{n,n}, v_{n-1,1}, v_{n-1,2}, ..., v_{n-1,n-1}, v_{n-2,1}, ..., v_{2,1}, v_{2,2}$ by the numbers:

CHAPTER 5. DIVISOR CORDIAL LABELING FOR SOME TREES AND FAMILIES OF GRAPHS

Where $(2m-1) \cdot 2^{k_m} \leq n$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1) \cdot 2^a$ divides $(2m-1) \cdot 2^b$; (a < b) and $(2m-1) \cdot 2^{k_i}$ does not divide 2m+1.

We will discuss the labeles of the vertices $v_{i,1}$ for the branches n-1, n-2, ..., 2 in two cases :

Case 1: If $f(v_{i,1})$ is odd then the edges $v_0v_{i,1}$ are labeled as 0

Case 2: If $f(v_{i,1})$ is even then the edges $v_0v_{i,1}$ are labeled as 1

That mean the labeling is same as chain and hold $|e_f(0) - e_f(1)| \leq 1$.

Example 5.4. The divisor cordial labeng of olive tree ol_6 is shown in the Figure 5.5.



Figure 5.5: A divisor cordial labeling for olive tree ol_6

5.3.2 Spider Tree

A tree is called a spider if it has a centeral vertex c with degree k > 1while each of the other vertices is either a leaf or has degree 2. Thus, a spider is an amalgamation of k paths with various lengths. If it has x_1 paths with length a_1, x_2 paths with length a_2 , and so on, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, ..., a_m^{x_m})$ where $x_1 + x_2 + ... + x_m = k$, any one of the paths from c to a leaf of T is called leg of the spider T. (see Figure 5.6).

Proposition 5.5. Spider Trees are divisor cordial graphs.

Proof. Let $SP(a_1^{x_1}, a_2^{x_2}, ..., a_m^{x_m})$ be a spider tree. Define the labeling of vertices by: label the center vertex as 2 and the leaf of the latest leg as 1 and proceeding with other vertices in this leg and other legs from long to short in the same way as in olive tree (proof of Proposition 5.3).

Example 5.6. The divisor cordial labeling for the spider tree $SP(4^2, 3^3)$, shown in Figure 5.6

5.3.3 m-stars Tree

An m - star tree has a single root node with any number of paths of length m such that each path attached to the root nod.

Proposition 5.7. All m – star trees are divisor cordial graphs.

Proof. Same as proof of Proposition 5.5

CHAPTER 5. DIVISOR CORDIAL LABELING FOR SOME TREES AND FAMILIES OF GRAPHS



Figure 5.6: A divisor cordial labeling for spider tree

Example 5.8. The divisor cordial labeling for 4-star tree.



Figure 5.7: The 4-star graph and its divisor cordial labeling

5.3.4 k-distant tree

A k-distant tree consists of a main path called the spine, such that each vertex on the spine is joined by an edge to at most one path on k-vertices. Those paths are called tails (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform k-distant tree. A uniform k-distant tree with odd number of vertices is called a uniform k-distant odd tree. A uniform k-distant tree with even number of vertices is called a uniform k-distant even tree. See Figure 5.8

Conjecture 1. The uniform k-distant tree is divisor cordial.



Figure 5.8: A divisor cordial labeling for the k-distant tree

5.3.5 Caterpillar Tree

Definition 25. A caterpillar is a tree T such that for a maximum path P, all vertices are of distance at most one from P. Figure 5.9

CHAPTER 5. DIVISOR CORDIAL LABELING FOR SOME TREES AND FAMILIES OF GRAPHS

Conjecture 2. All caterpillar trees are divisor cordial graphs.



Figure 5.9: Caterpillar tree

5.3.6 Banana Tree

A banana tree is constructed by bringing multiple stars together at a single vertex by an edge from each one. Figure 5.10

Conjecture 3. Banana trees are divisior cordial graphs.



Figure 5.10: Banana tree

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Appendix A

All Nonisomorphic Graphs with 7 Vertices and its Divisor Cordial Graphs

APPENDIX A. ALL NONISOMORPHIC GRAPHS WITH 7 VERTICES AND ITS DIVISOR CORDIAL GRAPHS

Connected graphs with 7 nodes

1 03 04 2 03 04 1 05	2 4 6 2 4 6 7 1	2 3 7 4 7 6	4 4 6 2 4 5 3 4 6	5 2 4 6 3 4 5 3 4 7		2 6 5	8 4 6 2 4 6 5 4 6	2 4 4	10 6 2 4 6 5 3	2^{11} 4^{3} 4^{6}			14 7 2 4 6 5 1	15 7 2 4 5 6 7
7 6 1111116	•5 •3 2111115	5 1 1111134	7 1 2111124	1 6 2111133	5 •3 2211114	7 1 2111133	2211123	2211123	7 1 2221113	1 • 7 2221122	5 7 2111126	1 7 2111135	7 3 2211125	3 1 2111144
16 4 7 2 4 6		18 14 7 2 4 4 5	2 3 4	20 4 6	246	2 4 7	23 4 6	24 4 6	2 45 4	2 4 5	2 4 1	28 4 6	29 7 4	$2 + \frac{1}{2}$
2211134	2211215	2111234	7 1 2211134	2211224	7 1 2221124	6 1 2211233	2221133	3111224	6 3 2211224	7.3	5 7 3111224	7 3 3211124	6 1 2211233	1122323
31 4 3 7 5 5	32 7 2 4 1	33 7 2 4 6	34 7 2 5 3	35 7 2 5 7	36 7 2 4 5 4	37 4 7 2 4 6	38 7 2 5 4	39 7 2 4 5 4	40 7 2 4 3	41 7 2 4 6	42 7 2 4 6	43 7 2 4 6	44 7 2 15 3	45 4 8 2 4 1
2 7 6	5 6 3	3070-5	6 4 7 4 1	3 6 1	7 3 6	7 X 3	7 3 1 6	7 1 3 6	6 7 × 1 5	3 5 17	3 7 5 1	37.115	1-7-64	6753
46 4 2 4 2 4 3	47 4 2 4 2 4 5	48 8	49 8	50 8 2 4 3	51 4 8 2 4 3	52 8 2 4 7	53 4 8 2 4 3	54 4 2 4 2 4 54	55 8 2 4 6	56 8 246	57 8 2 4 1	58 8 2 4 3	59 8 2 4 6	60 8 2 4 7
6 7 5	3 - 6	1 5 7	6 5 1	6 7 5 • 1	6 50 01	3 6	6 1 5 • 7	6 1 37	3 5 1	3 1 5	3 6 7	6 1	3 5 7	3 6 5
2211145 61 8	2221135 62 3	2221144 63 8	2211226 64 8	2211235 65 4 8	2221135 66 8	2221225 67 4 8	2211244 68 4 8	2221144 69 4 8	2221234 70 4 8	2222134 71 4 8	3111325 72 4 8	3211135 73 8	3211225 74 8	3111334 75 4 8
					2 6 3 5 7 5 1									
3211234 76 8	2211244 77 8	2221234 78 8	3111244 79 8	3211144 80 8	3211234	3221134 82 8	2121334 83 8	2122324 84	2211334 85 8	2221324 86 8	2222224 87 8	2211334 88 8	2221324 89	3111334 90 8
		2 4 6 5 4 7	$2^{2} + 4^{3} + 4^{4}$				2 6 5 3 4 1	2 3 7 1 5	2 6 1 3 5		2 4 6 7 3 5			
3121324 91 8	3211324 92 8	3221224 93 8	2122333 94 8	2221333 95 8	2222233 96 8	3211324 97 8	3311224 98 8	3311233 99 8	2222215 100 8	3212224 101 8	3213124 102 8	2222224 103 8	3211234 104 8	3212224 105 8
	2 6 5 7	2 4 6 3 4 5		2 4 6 3 5	2 45 1 3 4 4	2 4 7 4 3	2 4 6 3 7	2 4 3 6 4 1	2 4 1 6 4 3	2 4 6 3 4 1	2 4 4	2 4 5 4 7	2 4 5 6	
3221134	3221224	3222124	2222233	3312133	2222233	2322223	2331223	3113233	3212233	3213223	3312223	2321323	2322223	3221233
			2 4 6 3 4 6									2 4 6 5 4 6	2 4 1	
3221323	3222223	3311233	3311323	3312223	3321223	2221146	2221155	2222145	2221236	2221245	2222145	2222235	3 •5 3111336	3211236
	2 4 7 6 4 1	2 4 6 3 4 1					2 4 5		2 4 6 3 4 5	2 6 5		2 7		2 4 1 3 4 7
3111345	3211245	3211335	2221245	3211245	3221145	3221235	3211344	2222244	3221244	3222144	2221335	2222325	2121444	2122434
				2 4 3 5 6 1 7					² 4 6 3 7	2 4 6				
2211444 151 9	2221434 152 9	2222334 153 9	2221335 154 9	3211335 155 9	3221325 156 9	2221434 157 9	3111444 158 9	3121434 159 9	3211434 160 9	3221334 161 . 9	2222244	2222334 163 9	3122244 164 9	3122334 165 9
			2 4 6 7 1 3	2 4 6 3 7 7	2 4 6 3 7 1	2^{2} 4^{5}_{3} 6^{7}_{7} 6^{1}_{1}	2 4 3 6 7 5	2 4 6 3 7 5	$2^{2} + 6^{4}_{3} + 6^{7}_{5}$					$^{2}_{6}$ $^{4}_{7}_{7}_{3}$
3211344 166 9	3221244 167 9	3221334 168 9	3222234 169 9	3311325 170 9	3311334 171 9	3221334 172 9	3311244 173 9	3311334 174 9	3321234 175 9	2222226 176 9	3213135 177 9	3213225 178 9	3221235 179 9	3222135 180 9
		2 4 5 6 7 4 1						2 4 6 3 4 1	2 4 6 3 5 7					2 4 1 3 5 5
3222225	2222244 182 9	3212244 183 9	3213144 184 9	3213234 185 9	3222234 186 9	3223134 187 9	3213234 188 9	3312234 189 9	3313134 190 . 9	2222244 191 9	2322144 192 9	3212244 193 9	3222144 194 9	3222234 195 9
$ \frac{2}{6} \frac{4}{1} \frac{3}{7} \frac{3}{5} $			2^{2} $4^{6}_{7}_{7}_{7}_{1}_{5}^{7}_{5}$	2 7 5 5 1 1 3		2^{2} 4^{6}_{5} 3^{7}_{7} 1^{3}_{1}			2 4 3 5 6 7 1	2 + 4 = 6 = 7 + 5 = 1		$2^{2} + 4^{1}_{5} + 6^{4}_{3}$	2 4 1 3 6 5 7	2 4 7 5 4 3 6 1
3312144 196 9	3312234 197 9	3321144 198 9	3322134 199 9	2322324 200 9	2331324 201 9	2332224	3213333 203 9	3222333 204 9	3312333 205 9	332223 206 9	2331333 207 9	2332233 208 9	3222333 209 9	3321333 210 9
2 4 6 3 5 1 7		2 4 3 6 5 7			2 4 6 1 5 7	2 4 6 3 5 7 1				2 4 1 3 6 7 5			2 4 6 3 5 7	
3331233 211 9	3113334 212 9	3213234 213 9	3213324 214 9	3222234 215 9	3223224 216 9	3312324 217 9	3313224 218 9	2322324 219 10	3222234 220 10	3222324 221 10	3311334 222 10	3312234 223 10	3312324 224 10	3321234 225 10
2^{2} 4^{4}_{5} 6_{7}_{7}				2 4 3 1 5 7 6	2 4 6 1 3 5 7	2 + 4 = 6 1 + 5 + 7				2 6 1 0 3				
3321324 226 10	3322224	3222333 228 10	3322233 229 10	3233223 230 10	3323223 231 10	3332223 232 4 10	3332223	2222156 234 10	2222255 235 _4 10	2222246 236 10	3211346 237 10	3221246 238 10	3211355 239 10	3221345 240 10
			2 6 3 5 7											
2222255 241 _4 10	3221255 242 10	3222155 243 4 10	3222245 244 10	2222336 245 10	2221445 246 10	2222435 247 10	3221336 248 10	2221445 249 10	3211445 250 10	3221435 251 10	2222345 252 10	3122345 253 10	3221345 254 10	3222245 255 10
		6 3 7 5	6 3 7			6 5 7				2 4 6 3 5 7	6 5 7 4			
3222335	3222344	3311336	3211445	3311435	3221345	3311345	3321245	3321335	3321344	3322244	2222444	3221444	3222434	3311444

Figure A.1: A divisor cordial labelin g for all connected graphs with seven vertices

APPENDIX A. ALL NONISOMORPHIC GRAPHS WITH 7 VERTICES AND ITS DIVISOR CORDIAL GRAPHS

256 4 10	257 10	258 4 10	259 10	260 10	261 10	262 10	263 10	264 4 10	265 10	266 4 10	267 10	268 10	269 10	270 10
	2 3 5 6 7	2 5 6 3 5 1 7		2 4 1 5 6 7 3	6 5 7									2 3 7 7
3312434	3321434	3322334	3221444	3331334	3213236	3222236	3213245	3222245	3223145	3223235	3213245	3312245	3313145	3313235
2 4 1	2 4 3	2 43	2 4 6	2 4 7	2 4 6	2 4 6	2 4 3	2 1 4	2 1 4	2 1 5	2 1 4	2 1 4	2 1 4	2
6 7 3	6 4 1 7	6 71	5 2 7 1	3 5 6	3 1 5	3	6	6 3 3	5 3	4	6 3 3	3	3 5	3
3222245	3312245	3322145	3322235	3223244	3313244	3322244	3323144	2331335	2332325	2222444	2321444	2322434	2331434	2332334
286 10	287 10	288 10	289 10	290 10	291 10	292 10	293 10 2 1 4	294 10 2 1 4	295 10	296 10	297 10 2 4 1	298 10	299 10	300 10
4 17	6	3 1 17	6 57	3 51	3	4	6	6	1 3	6 07	6 5	6 51	3	6 4 3
6 5 3113444	5 3 3213344	6 5 3213434	3 5	5 3222434	3223334	6 5 2222444	2312444	3 5	6 5 2322434	3 • •5	3 • • / 3221444	3 5 3222344	6 5 3222434	5 • 7 3311444
301 4 10	302 4 10	303 10	304 10	305 4 10	306 10	307 10	308 4 10	309 4 10	310 10	311 1 10	312 1 10	313 4 10	314 4 10	315 4 10
6	3 At 17	6 K1	5 K 1	6 K 1	7883	6	3 2 2 1	3 8 8 1	3	6	545	6	5 4 1	6 37
5 1	6 1	5 7	3 7	5-2-2	6 5	1 5	6 5	5 7	5 7 7	5- 7	6 7	5 7	6 7	1 5
316 10	317 10	318 10	319 10	320 10	321 10	322 10	323 10	324 10	325 10	326 10	327 10	328 10	329 10	330 10
2 1 1 5 1 7	2 6	2 A 1	2 A 3	2 6	2 A 6	2 A 3		2 A 3		2	2 A 1	2 A 3	2 6	$\frac{2}{3}$
6 3	5-01	547	5-7	5 7	30 5	5 57	5 .7	6 7	5 3	1 5	6 7	64-1	1005	5 57
3331334 331 10	3332234 332 10	3213335 333 10	3223235 334 10	3223325	3313325 336 10	3321335 337 10	3322235 338 10	3322325 339 , 10	2323244 340 10	3313244 341 10	3313334 342 10	3322334 343 10	3323234 344 10	4222334 345 . 10
2 A 1	2 45 4	2 A 1	2 A 1	$^{2}A^{3}$	2 A 4	2 A 3	2 A 1	2 A 3	2 3	2 4 6	2 46	2 A 4	2 4 6	2
° ¥31,5'	3 6 V 1	· V3 5	644	5 1	617	617	5 3	6 1	6V 7	5 7	3 3 1	6 47	5 77	1 15
3212444 346 10	3312434 347 10	4211444 348 10	3313334 349 10	4312244 350 10	3322334 351 11	4222334 352 11	4311344 353 11	4312334 354 , 11	4321334 355 11	4322234 356 11	3332333 357 11	2333324 358 11	3323324 359 11	3333224 360 11
$^{2}A^{3}_{1}$	2.46	2	2	2	2 4	2	2	2	2	2	2	2 A 1	2	
647	123	5 3	1	1 6	5 7	5 7	5 7	6 7	5 7	5 7	5 7	5 7	3 5	5 .7
3323333 361 , 11	3333233 362 , 11	3332324 363 11	3333224 364 11	3333233 365 11	2222266 366 11	3221356 367 11	3222256 368 11	3222355 369 11	2222446 370 11	3221446 371 , 11	3222346 372 , 11	2222455 373 11	3122455 374 11	3221455 375 , 11
$2 \mathbb{A}^{1}$	2 2 6	2	2	2 4	2 A 4	2 4	2	2	2	2	2	2 3	2	2
547	3 1	5 7	3 7	5 3	5 .7	6 3	5 7	6 7	5 3	5 7	6 7	5 7	6 67	6 7
3222355 376 . 11	3222445 377 11	3311446 378 . 11	3321346 379 11	3311455 380 . 11	3321355 381 11	3321445 382 . 11	3322255 383 . 11	3322345 384 . 11	3222445 385 . 11	3321445 386 . 11	3322435 387 . 11	3321445 388 , 11	3331435 389 , 11	3322444 390 . 11
2 4 4	2 4 6	2 4 1	2 3	2 4	2 4 3	2	2 3	2 3	2 3	2 6	2 4	2 4	2 4	2 4
5 6 3 '	³ 5×1	5 7	125	5 6 X -3	5 11	5 3	105	6 1 5 T	5 77	5 7	5 A7	5 1 3	5 77	5 6 7 3
3331444 391 11	3332344 392 11	3223246 393 11	3313246 394 11	3313345 395 11	3322246 396 11	3223255 397 11	3313255 398 11	3322255 399 11	3323155 400 11	3323245 401 11	2332336 402 11	2322445 403 11	2331445 404 11	2332435 405 . 11
2 4 1	2 4 1	2 4 1	2 4 1	2 4	2 4 1	2 4 4	2 4	2 4 3	2 6	2004 1	2 14	2 3 3	2 3	2 4
6 4 3	6 A 75	65 37 5 37	5 6 7 3	6 3 3 7 5 7	6 3 2 7 7 3 7 5 7	6 5 4 3 7	65 A37	5 A7	3 5 7 7 1	6 5 7 5 3 7	6 3 × 5 7	5 A7	65 7 5 7	³ 6 5 7
3213445 405 11	3222445 407 11	3223345 408 11	3223435 409 11	2322445 410 11	3222445 411 11	3312445 412 11	3321445 413 . 11	3322345 414 11	3322435 415 . 11	3223345 416 . 11	3313345 417 11	3322345 418 . 11	3323245 419 11	3323335 420 . 11
2 1 4	2 4 4	2 3	2 2 3	2 3	2 6	2 1 4	2 6	2 4 1	2 4 3	2 4 1	2 3	2 4 4	2 4 3	2 43
57×63	6 5 7 3	5 7	6 5 P7	6 5 A 1 7	3 5 7 7 5 1 1	5 36	³	5 7	645	6 3 4 5 T	6 5 V7	6 3 4 5 T	5 7	5 A7
3323344 421 11	2332345 422 11	3322345 423 11	3331345 424 11	3332245 425 11	3332335 426 11	3332344 427 11	3333244 428 11	3223336 429 11	4114435 430 11	3313336 431 11	3322336 432 11	3313345 433 11	3322345 434 11	3323245 435 11
2 4 6	2 1 1	2 4 1	2 34	2 4 1	2 1	2 4 1	2 4 1	2 4 1	2 4 1	2 3	2 3	2 4 6	2 4 4	2 A7 1
3 5 Tr	6 5 7 3	6 5 T 3	6 5 V 1 7	6 3 5 7	6 5 V 3 7	5 6 X 3 7	6 4 7	6 5 4 7 ³	5 3 6 6	5 57	6 5 V7	3 5W7	5 6 7	5 3
3323335 436 11	4221445	4222435	3312445 439 11	4212445	4311445	4312435	4312345 443 11	4313245	4313335	4321345	4322245 447 11	4322335 448 11	3323344 449 11	3223444 450 11
2 4 5	2 4 1	2 1 1	2 1	2 31	2 1 1	2 43 4	2 4 4	2 1	2004-1	2	20041	2 1	2	2 47
3 6 1 7	5 6 3 1	6 5 - 3 5 - 7	4 6 5	4 6 5	4 6 57	6 5 1 7 5 1	6 5 A 3	3 6 5 4	6 5 47 3	6 5 3 5 7	6 5 A7	5 6 37	⁶ 3 4 5	3 6 4 1 5 6 4 1
4114444	4124434	4213444	3313444	3223444	3313444	3322444	3323344	3323434	4213444	4222444	4223344	4223434	4313344	4313434
2 4 3	2 4 3	2 4 4	2 41	2 4 3	2 41	2 4 4	2 3 4	2 4 4	2 4 4	2	2 2 3	2 4 1	2 4 6	2 46
6 5 A7 1	5 6 7	6 5 7 3	6 5 7 3	6 5 7 1 5 7	5 7 3 6	57836	6 5 7 1 5 7 7	6 3 5 T	6 5 P 7	6 5 7 3	6 5 47 1 5 47	3 6 5	3 5 A 1	1.50737
4322344	4322434	2332444	3223444	3322444 470 11	3332434	3333334 472 11	3322444 473 11	3331444 474 11	3332434 475 11	4222444	4321444	4331434	4332334	3333325 480 11
2 4 4	2	2 41	2 4 6	2 4 4	2 131	2 4 17	2 4 4	2 43	2 4 3	2 43	2 43	2 43	2 4 6	2 41
5 6 3 3	6 5 3 7 3	5 6 7 3	3 5 7 1	5 6 3 7	5 6 4 7	⁶ 3 5	3 6 4 5 6 4 1	6 1 1 7 7 1 1 5 7	125	6 5 47 1	6 5 7 7 5 1 1	6 1 5	1 5 37 5 37	3 6 57
2334334 481 . 11	3234244 482 11	3234334 483 11	3324334 484 11	3333334 485 11	3234334 486 11	4133344 487 11	3333334 488 11	4233244 489 12	4323244 490 12	4323334 491 12	4332244 492 12	4333234 493 12	3333325 494 12	4241434 495 12
2	2	2 33	2	2	2	2	2 4 5	2 4 4	2 1	2 4	2 4	2 4 4	2 4 4	2017
⁵ 6 3 7	6 5 A7	5 17	5 6 1 3	5 1 67	5 77	5 6 7 3	1 73	5 37	5 × 7	5 7	5 7	⁶ 5 7 3	5 37	5
3333334 496 12	4323244 497 12	4332244	4332334 499 12	4333234 500 12	4242244	4242334 502 12	4341244 503 12	3222366 504 12	3222456 505 12	3321456 506 12	3322356 507 12	3322455 508 12	3322446 509 12	3222555 510 12
2 3	2	2 2 4	2 4 4	2	2	2 1 4	2 4	2	2 4	2	2 4	2 4	2 4	2007
°6₩~4	5 × 7	5 73	3 6 7 3	⁰ 5 7 3	° 6 7 3	° 3 8 6	3475	5 7 3	5 × 7	°5₩73	5 × 37	3 A57	5 37	5 43 ⁴
3322545	3331446	3321555	3331545	3332355	3332445	3333444	3313356	3323256	3323355	2332446	3223446	3322446	3323346	3223455

Figure A.2: A divisor cordial labeling for all connected graphs with seven vertices

APPENDIX A. ALL NONISOMORPHIC GRAPHS WITH 7 VERTICES AND ITS DIVISOR CORDIAL GRAPHS

511 .5 12	512 12	513 12	514 12	515 .5 12	516 12	517 12	518 12	519 4 12	520 1 12	521 .4 12	522 6 12	523 12	524 1 12	525 12
	2 4					2 3 6					1 37	6 1 3		
3313455 526 12	3323355	3323445 528 12	3332346 529 12	2332455	3322455	3331455 532 12	3332355 533 12	3332445 534 12	3333255	3333345 536 12	4114446	4114455	4214445	4214355
2 6	2 6	2 6	2 4 6 4 4 6 7	2 4 6 4 4 4 3	2 4 6 4 4 4 3	2 A 6	2 A 6	2	2 4 6 4 6 3	2 4 3 4 4 6	2 6	2 4 6 4 4 4 3	2 A 6	2 A 6
5 7 4223355	5 7	5 7	3323346	5 7	4312446	5 7 4313346	4313445	4322346	3322455	4222455	4312455	4321455	4322355	4322445
541 12 2 A 6	542 12 2 A 6	543 12 2 A 6	544 12 2 A 6	545 12 2 A 6	546 12 2 1 6	547 12 2 A 6	548 12 2 1 3	549 12 2 1 6	550 12 2 A 6	551 12 2 A 6	552 12 2 1 3	553 12 2 A 6	554 12 2 A 6	555 12 2 A 6
4 3 5	4 4 -3 5 7 7	4 3 7	5 4 3 4 7	4 3	4 3 3	5 4 7	4 5 6 7	4 3 3	4 5 7	4 3 5 7	4 5 7 6	4 5 7	4 5 3	4 3 3
3323355 556 12	4313355 557 12	4323255 558 12	4323345 559 12	4214445 560 12	4223445 561 12	4224435	4214445 563 12	4313445 564 12	4314435 565 12	3323445 566 12	4223445	4313445 568 12	4322445 569 12	4323345 570 12
2 6 5 4 7 3														
4323435 571 12	3323445 572 12	3332445 573 12	3333435 574 12	3332445 575 12	4322445 576 12	4331445 577 12	4332435 578 12	3333444 579 12	3333336 580 12	3234345 581 12	3324345 582 12	3334245 583 12	3334335 584 12	2334444 585 12
						2 4 6 5 4 7 3		2 5 6		2 4 3 5 7 6		$\frac{2}{5}$ $\frac{1}{7}$ $\frac{6}{4}$ $\frac{3}{4}$	2 4 6	247 43765
3324444 586 12	4134345	4233345	4234245	4234335	4333245	4333335	3324444 593 12	3334344 594 12	3334434 595 12	4224444	4234344	4334334	3324444 599 12	4314444
								2 3 3 5 7						
4324344	4414344	4414434	4423434	4424334	4422444	4432434	4433334	3333336	4241445	4242435	4333245	4333335	4242345	4332345
2000	2	2 4 6	1 A 3	2 3 3 S	2 A X 6	2 XX 6	2	2	2 4 1 T	2	2	2 4 6	2 4 6	2
5 7	5 7	4 7	5 7	5 7	5 7	4333434	4 7	5 7	4 5	5 7	4 7	4 7	7 4	3322466
616 13 2 1 6	617 13 2 1 6	618 13 2 4 3 6	619 13 2 7 6	620 13 2 4 6	621 13 2 1 6	622 13 2 3 6	623 13 2 5 6	624 13 2 4 6	625 13 2 4 6	626 13 201 6	627 13 2 4 6	628 13 2 1 6	629 13 2 1 6	630 13 2 4 6
4	4	4 1 7	5 1 3	1 5 3 5 7	4 3 5 7	4	4	4 3 5	5	4 5 3	4 5 7	5 4 7	4 5 7 3	4 5 7 3
3322556 631 13	3331556 632 13	3332456 633 13	3332555 634 13	3333545 635 13	3323366 636 13	3323456 637 13	3332456 638 13	3333356 639 13	3333455 640 13	4214456 641 13	4224356 642 13	4224455 643 13	4313456 644 13	4322456 645 13
2 4 5 7 6 3													2 6 4 5 7 7	
4323356 646 13	4323455 647 13	4224446 648 13	4314446 649 13	4214555 650 13	4314545 651 13	4323446 652 13	4224455 653 13	4324355 654 13	4324445 655 13	3333446 656 13	3323555 657 13	4223555 658 13	4313555 659 13	4323455 660 13
					2 6 5 4 7 3			2 1 5 7 7						4^{2}_{5}
4323545 661 13	4332446 662 13	3332555 663 13	4322555 664 13	4331555 665 13	4332545 666 13	3333455 667 13	4333355 668 13	4333445 669 13	3334346 670 13	2334455 671 13	3324455 672 13	3334445 673 13	4234346 674 13	4224455 675 13
								$2^{2}_{5}_{4}_{4}_{7}^{6}_{7}_{7}^{6}_{3}$			245-3 45-7-3		2 5 4 5 4 6 3	
4314455 676 13	4324445 677 13	4333346 678 13	3334355 679 13	4234355 680 13	4324355 681 13	4333355 682 13	4334255 683 13	4334345 684 13	3334445 685 13	4324445 686 13	4334345 687 13	4334435 688 13	4414445 689 13	4324355 690 13
		4 4 3			4 4 3		2 5 5 7 3					2 5 7 4 3		
4414355 691 . 13	4423355 692 13	4424255 693 13	4424345 694 13	4324445 695 13	4414445 696 13	4424345 697 13	4424435 698 13	4432445 699 _ 13	4433435 700 13	4433444 701 13	4434434	4444334 703 . 13	4242446 704 13	4333346 705 13
2 4 5 °	2 3 4 3 6	2 4 6 4 4 5 1	2 1 6 5 1 6	2 1 6 4 1 5 2	2 4 6 1 6 1 1 C	2 4 6 4 4 3	2 5 6 5 5 5 2	2 × 5	2 4 6 3	2 6 5 6	2 5 6 1 C	2 5 5 5 1 1	2 16 5 16	2 5 1 6 5 1 3
5 7 4342346	5 7	5 7 4341455	4342445	5 7 4333355	5 7 4342355	5 7 4343255	4343345	4333445	5 7 4342445	4343435	5 7 4432445	4433435	4433444	4442444
706 13 2 6	707 13 2 13 6	708 13 2 1 6	709 13 2 A 6	710 14 4 3 6	711 14 2 4 6	712 14 2 6 7	713 14 2 1 7	714 14 2 4 6	715 14 2 1 6	716 14 2 1 6	717 14 2 1 6	718 14 2 3 6	719 14 2 16	720 14 2 4 6
57343	5 4 7	5 4 7 3	5 7 4 3	2 5	5 3 3 1 7 7	4	4 3 6	4 3 5 7	4 3 5 7	4 3 5	4 3 5 7	4	4 3 5 7	4 37 3
721 14	722 14	723 14	724 14	725 14	726 14	727 14	728 14	729 14	730 14	731 14	732 14	733 14	734 14	735 14
4 3 5 7	4 3 5 7	4 3 5 7 4 3 5 7	4 3 5 7	4	4 5 37	4 5 3 7	3 4 5 7	2 4 5 7	2	4 5 37	4 5 37	4 5 3 7	3 4 5	4 3 5 7
4333456 736 14	4324555 737 14	4333555 738 14	4334545 739 14	3334456 740 14	4324456	4334356 742 14	4334455 743 14	3325555 744 14	3335545 745 14	4334446 746 14	4334455	4335355	4335445 749 14	4414456 750 14
4 3 5 7	3 4 5 7	645 3	4 3 5 7	5 4 5 7	6 4 5 4 3	5 4 3 7			3 4 5 T	4 3 5 7	4 3 5 7		3 4 5 7	4 3 5 7
4424356 751 14	4424455 752 14	4225555 753 14	4325455 754 14	4325545 755 14	4315555	4415545	4424455 758 14	4425355 759 14	4425445	4424446 761 14	4433446	4432555 763 14	4433545	4434355
3^{2}_{4} $5^{6}_{7}_{5}$								2 5 4 5 4 7						2 3 4 5 7
4434445	4444435	4444444	4342456	4343356	4343455	4343446	4333555	4343545	4433446	4433545	4342555	4432555	4441555	4442545

Figure A.3: A divisor cordial labeling for all connected graphs with seven vertices

APPENDIX A. ALL NONISOMORPHIC GRAPHS WITH 7 VERTICES AND ITS DIVISOR CORDIAL GRAPHS

766 14	767 14 2 1 6	768 14	769 14	770 14	771 14	772 14	773 14	774 15	775 15 2 1 6	776 15	777 15 2 A 6	778 15 2 1 6	779 15 2 13 6	780 15
4 5 7	3 4 5	4 7 7	4 7 7	4 3 5 7	3 4 5 7	3 4 5	3 4 5 7	4 5	4 3 5 7	4 5 7	4 3 5 7	4 5 3	1457	4 7
4443355	4443445	4434355	4443355	4444255	4444345	4444435	4444444	3333666	4324566	4333566	4334556	4334466	3335556	4335456
781 15	782 15	783 15	784 15	785 15	786 1 15	787 15	788 15	789 15	790 15	791 15	792 1 15	793 15	794 15	795 15
² 4 3 5 7		² 4 3 5 7	² 4 3 5 7	² 4 3 5 7	² 4 3 7 7	4 3 5 6	² 4 3 3 5	² 4 3 3 5 7	² 3 4 5 7		² 4 3 5 7	1 4 6 2 4 5 7	² 4 5 5 3 7	² 4 3 5 7
4424466	4325556	4415556	4425456	4433556	4434456	4335555	4425555	4434555	4435545	444446	4445355	4445445	4343466	4343556
796 1 15	797 15	798 15	799 15	800 15	801 15	802 15	803 15	804 15	805 15	806 .4 15	807 15	808 15	809 15	810 15
			² 4 3 5 7	$^{2}_{3}$ $^{6}_{4}$ $^{7}_{5}$	2 0 0 7 3 0 5	2 6 1 5 7 7				2^{2} 6^{1} 3^{2} 5^{7}	² 4 3 5 7	² ³ ⁴ ⁶ ⁷	² 4 3 5	
4433556	4442556	4443456	4443555	4444545	5155545	5255355	5255445	5353455	5354355	5354445	4444356	4444455	444446	5424555
811 15 2 4 6 1 5 3 7	812 15 2 4 6 3 5 7 1 5 5	813 15 2 4 6 1 5 3 7	814 16 2 4 1 5 7	815 16 2 4 1 3 5	816 16 2 7 6 4 7 5 3		818 16 2 7 6 4 5 1 3	819 16 2 7 6 4 3 1 5	820 16 2 7 6 1 4 3 5	821 16 2 7 6 4 7 5 3	822 16 2 7 6 4 7 6 1 3	823 16 2 1 7 4 3 5	824 16 2 4 7 1 3 5 5	825 16 2 6 7 1 4 3 5
811 15 2 4 6 1 5 3 5434545 2 6 1 7 5434545 1 6 1 7 5434545	812 15 2 4 6 3 5 5443455 827 16	813 15 2 4 6 1 5 3 5444445	814 16 2 4 4 4 4 3 4 5 4 3 4 5 4 3 4 5 4 3 4 5 4 5 4 5 4 5 4 5 4 5 4 5 4 5 4 5 4 5 6 6 7 7 6 4 5 7 6 7 7 6 4 5 7 7 7 7 7 7 7 7 7 7 7 7 7	815 16 2 7 6 4 5 4 335566 820 16	816 16 2 4 4 4 4 4 4 4 5 6 4 5 3 4 4 5 6 8 16 16 16 16 16 16 16 16 16 16	817 16 2 7 6 4 5 4434566 822 16	818 16 2 4 4 4 4 4 4 4 4 4 4 4 5 6 4 5 1 6 4 5 1 6 1 7 1 6 1 7 1 6 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 7 1 1 1 1 1 1 1 1 1 1 1 1 1	819 16 2 7 6 4 3 5 4445456 824 16	820 16 2 7 6 1 7 5 4445555 825 17	821 16 2 4 4 4 4 4 4 4 4 4 4 5 6 17 6 5 16 5 16 16 16 16 16 16 16 16 16 16	822 16 2 4 4 4 4 4 4 4 4 4 4 4 4 4	823 16 2 7 4 3 5 5155556	824 16 2 4 7 1 3 5 5255456 20 17	825 16 2 6 7 1 4 3 5353556 840 17
811 15 2 4 6 1 5 5434545 826 16 2 4 6 5434545 826 16 2 4 6 7 5 5434545 826 7 7 6 7 7 5434545 826 7 7 7 5434545 826 7 7 8 7 8 7 8 7 8 7 8 7 8 7 8 7	812 15 2 4 6 3 5443455 827 4 6 1 5 5443455 827 4 6 1 5 7 3 5 7 3 5 7 1 5 7 3 5 7 4 6 7 3 5 7 4 6 7 1 5 7 4 6 7 1 5 7 1 7 1 5 7 1 6 7 1 7 1 7 1 7 1 7 1 7 1 7 1 7 1	813 15 2 4 6 1 5 5 444445 828 16 2 4 6 5 444445 828 16 2 4 6 7 5 5 444445 8 7 7 5 7 7 5 444445 8 7 7 7 5 444445 8 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7	$\begin{array}{c} 814 & 16 \\ 2 & 3 & 6 \\ 4 & 5 & 7 \\ 4 & 334566 \\ 829 & 16 \\ 2 & 16 \\ 4 & 3 & 5 \\ 7 & 3 & 5 \end{array}$	815 16 2 7 6 4 335566 830 16 2 6 4 335566 830 5 7 7 3 5 7	$\begin{array}{c} 816 & 16 \\ 2 & 7 & 6 \\ 4 & 3 & 5 \\ 4 & 425566 \\ 831 & 16 \\ 2 & 6 \\ 3 & 4 & 5 \\ 7 & 4 & 5 \end{array}$	$\begin{array}{c} 817 & 16 \\ 2 & 7 & 6 \\ 4 & 3 & 5 \\ 4434566 \\ 832 & 16 \\ 2 & 16 \\ 2 & 5 \\ 4 & 3 & 5 \end{array}$	818 16 2 7 6 4 5 1 3 4435556 833 16 2 7 3 5 7 3 5 7	819 16 2 7 6 4 3 5 4445456 834 16 3 4 5 7 7	820 16 2 7 6 1 4445555 835 7 17 2 7 6 4 445555 835 17 2 7 6 4 45555	$\begin{array}{c} 821 & 16 \\ 2 & 7 & 6 \\ 4 & 3 & 5 \\ 4443566 \\ 836 & 17 \\ 2 & 7 & 6 \\ 4 & 5 & 5 \\ 1 & 3 & 3 \end{array}$	$\begin{array}{c} 822 & 16 \\ 2 & 7 & 6 \\ 4 & 3 \\ 4444556 \\ 837 & 6 \\ 7 \\ 2 & 7 \\ 4 & 3 \end{array}$	$\begin{array}{c} 823 & 16\\ 2 & 7\\ 4 & 5\\ 5155556\\ 838 & 17\\ 2 & 7\\ 4 & 3 & 5\\ \end{array}$	$\begin{array}{c} 824 & 16 \\ 2 & 4 \\ 1 & 3 \\ 5255456 \\ 839 & 17 \\ 2 & 4 \\ 4 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\ 6 \\ 1 & 5 \\$	$\begin{array}{c} 825 & 16 \\ 2 & 6 \\ 7 \\ 5 \\ 5353556 \\ 840 & 17 \\ 2 & 6 \\ 4 \\ 3 & 5 \\ 7 \\ 7 \\ 7 \\ 3 \\ 5 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7$
$\begin{array}{c} 811 & 15 \\ 2 & 4 & 6 \\ 1 & 5 & 3 \\ 5 & 434545 \\ 826 & 16 \\ 2 & 4 & 6 \\ 4 & 3 & 5 \\ 5 & 5354456 \end{array}$	$\begin{array}{c} 812 & 15 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ 5 & 5 & 443455 \\ 827 & 4 & 6 \\ 1 & 5 & 7 \\ 4444466 \end{array}$	813 15 2 4 6 5 4 4 4 4 4 5 8 2 8 16 2 4 5 5 4 3 4 5 5 5 4 4 4 4 5 5 4 4 4 4 5 5 5 5 5 4 4 4 4 5 5 5 5 5 4 4 4 4 5 5 5 5 5 6 7 5 7 5 7 6 7 7 7 7 7 7 7 7 7 7 7 7 7 7	814 16 2 4 4 4 4 3 4 5 4 3 4 5 7 7 7 7 7 7 7 7 7 7 7 7 7	815 16 2 6 4 335566 830 16 2 6 4 335566 830 5 5 7 3 5 5 444555			818 16 2 7 6 4 3 4 435556 8 3 16 2 4 6 4 3 55 5 552555	819 16 2 7 6 4 3 7 6 4 3 7 6 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	820 16 2 7 6 1 7 6 4 445555 835 17 2 7 6 4 7 6 5 3 4 445555 835 17 2 7 6 4 435566	$\begin{array}{c} 821 & 16 \\ 2 & 6 \\ 4 & 3 \\ 4443566 \\ 836 & 17 \\ 2 & 6 \\ 4 & 5 \\ 4445566 \end{array}$	$\begin{array}{c} 822 & 16 \\ 2 & 7 & 6 \\ 4 & 1 & 3 \\ 4444556 \\ 837 & 6 & 7 \\ 2 & 7 & 5 \\ 4444566 \end{array}$	$\begin{array}{c} 823 & 16 \\ 2 & 7 \\ 4 & 5 \\ 5155556 \\ 838 & 17 \\ 2 & 7 \\ 4 & 5 \\ 5255566 \end{array}$	$\begin{array}{c} 824 & 16 \\ 2 & 4 & 7 \\ 1 & 3 & 5 \\ 5255456 \\ 839 & 17 \\ 2 & 4 & 7 \\ 839 & 17 \\ 2 & 5 \\ 5354566 \end{array}$	$\begin{array}{c} 825 & 16 \\ 2 & 6 & 7 \\ 1 & 4 & 3 \\ 5353556 \\ 840 & 17 \\ 2 & 6 \\ 4 & 3 & 5 \\ 5444566 \end{array}$
$\begin{array}{c} 811 & 15 \\ 2 & 4 & 6 \\ 1 & 5 & 7 \\ 5 & 7 & 7 \\ 5 & 7 & 7 \\ 5 & 7 & 7 \\ 826 & 16 \\ 4 & 5 & 7 \\ 3 & 5 & 7 \\ 3 & 5 & 7 \\ 3 & 5 & 7 \\ 3 & 5 & 7 \\ \end{array}$	$\begin{array}{c} 812 & 15 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ 5 & 5 & 443455 \\ 827 & 16 \\ 2 & 4 & 6 \\ 1 & 5 & 7 \\ 4444466 \\ 842 & 17 \\ 2 & 4 & 6 \\ 842 & 17 \\ 2 & 5 \\ 7 & 7 \\ 3 & 5 \\ 7 \end{array}$	$\begin{array}{c} 813 & 15 \\ 2 & 4 & 6 \\ 1 & 5 & 3 \\ 5 & 444445 \\ 828 & 16 \\ 2 & 6 \\ 4 & 3 & 5 \\ 5434556 \\ 843 & 17 \\ 2 & 4 & 6 \\ 843 & 17 \\ 2 & 5 \\ 7 & 3 & 5 \\ 7 & 7 \\ 7 & 5 \\ 7 & 7 \\ 7 & 5 \\ 7 & 7 \\ $	$\begin{array}{c} 814 & 16 \\ 2 & 3 & 6 \\ 4 & 5 & 7 \\ 1 & 5 & 7 \\ 4 & 3 & 4666 \\ 829 & 16 \\ 2 & 1 & 6 \\ 829 & 16 \\ 7 \\ 3 & 5 & 7 \\ 844 & 17 \\ 2 & 5 \\ 844 & 17 \\ 2 & 5 \\ 7 \\ 4 & 5 \\ 7 \\ 7 \\ 4 & 5 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\$	$\begin{array}{c} 815 & 16 \\ 2 & 7 & 6 \\ 4 & 35566 \\ 830 & 16 \\ 2 & 6 \\ 4 & 3455 \\ 845 & 7 & 6 \\ 845 & 18 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 2 & 3 \\ 1 & 7 & 6 \\ 1 & 7 & 6 \\ 1 & 7 & 6 \\ 1 & 7 & 7 \\ 1$	$\begin{array}{c} 816 & 16 \\ 2 & 7 & 6 \\ 4 & 3 & 5 \\ 4425566 \\ 831 & 16 \\ 2 & 6 \\ 3 & 442556 \\ 831 & 5 \\ 7 & 5 \\ 5445545 \\ 846 & 6 \\ 18 \\ 2 & 6 \\ 1 & 3 \\ 3 \\ 4 & 5 \\ 7 & 5 \\ 4 & 3 \\ 1 & 3 \\ 3 \\ 1 & 3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 5 \\$	$\begin{array}{c} 817 & 16 \\ 2 & 7 & 6 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 8 & 5 \\ 8 & 7 & 18 \\ 2 & 7 & 6 \\ 4 & 5 \\ 8 & 5 \\ 8 & 7 & 18 \\ 2 & 7 & 6 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 4 & 5 \\ 5 & 5 \\ 4 & 5 \\ 5 & 5 \\ 4 & 5 \\ 5 & 5 \\ 4 & 5 \\ 5 & 5 $	818 16 2 7 6 4 1 3 55 4 435556 833 16 2 7 6 833 16 2 7 6 8 35 5 555555 848 18 1 7 6 2 4 3 55 848 18 1 7 6 2 4 3 55 5 555555 5 555555 5 5 5 5 5 5 5 5	$\begin{array}{c} 819 & 16 \\ 2 & 76 \\ 4 & 3 \\ 4 & 445456 \\ 834 & 16 \\ 3 & 445555 \\ 849 & 18 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5$	$\begin{array}{c} 820 & 16 \\ 2 & 7 & 6 \\ 1 & 5 & 5 \\ 4445555 \\ 835 & 17 \\ 2 & 7 & 6 \\ 4 & 3 & 5 \\ 4435666 \\ 850 & 19 \\ 1 & 6 \\ 2 & 4 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 7 & 6 \\ 1 & 6 & 5 \\ 2 & 4 & 3 & 5 \\ 1 & 6 & 5 \\ 2 & 4 & 3 & 5 \\ 1 & 6 & 5 \\ 2 & 4 & 3 & 5 \\ 1 & 6 & 5 \\ 2 & 4 & 3 & 5 \\ 1 & 6 & 5 \\ 1 $	$\begin{array}{c} 821 & 16 \\ 2 & 76 \\ 4 & 33 \\ 4443566 \\ 836 & 17 \\ 2 & 76 \\ 4 & 556 \\ 851 & 19 \\ 1 & 76 \\ 851 & 19 \\ 2 \\ 4 & 3 \\ 7 & 6 \\ 851 & 36 \\ 7 & 86 \\ 851 & 36 \\ 7 & 86 \\ 851 & 19 \\ 7 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 851 & 19 \\ 85 & 86 \\ 85 & 10 \\ 85 & 1$	$\begin{array}{c} 822 & 16 \\ 2 & 7 & 6 \\ 4 & 3 \\ 4444556 \\ 837 & 17 \\ 2 & 4444566 \\ 837 & 5 \\ 4444666 \\ 852 & 70 \\ 1 & 7 & 6 \\ 2 & 3 \\ 7 & 6 \\ 2 & 3 \\ 3 \end{array}$	$\begin{array}{c} 823 & 16 \\ 2 & 4 \\ 3 & 5 \\ 5155556 \\ 838 & 17 \\ 2 & 4 \\ 3 & 5 \\ 525556 \\ 853 & 76 \\ 853 & 76 \\ 2 & 3 \\ 4 & 4 \\ \end{array}$	824 16 2 4 7 1 3 5 5 2255456 839 17 2 4 5 5 354566	$\begin{array}{c} 825 & 16 \\ 2 & 6 \\ 7 \\ 1 & 3 \\ 5353556 \\ 840 & 17 \\ 2 & 6 \\ 4 \\ 3 & 5 \\ 5444566 \end{array}$

Figure A.4: A divisor cordial labeling for all connected graphs with seven vertices