Republic of Iraq
Ministry of Higher Education
and Scientific Research
The University of Al-Mustansiriyah
College of Science

Error Estimate of Fourier Series by Moduli of Smoothness

A Thesis

Submitted to the Department of Mathematics, College of Science, The University of Al-Mustansiriyah as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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(B.Sc., 2012)

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September

2016

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الاهداء

بسم الله الرحمن الرحيم

(وقل إعملوا فسيرى الله عملكم ورسوله والمؤمنون)

اذا كان الاهداء ولو بجزء من الوفاء فالاهداء

الـــى . . .

معلم البشرية ومنبع العلم نبينا محمد (صلى الله عليه وسلم)

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الحب كل الحب . . . أخوتي وأخواتي

الـــى . . .

كافة الأهل والأصدقاء

ACKNOWLEDGEMENT

Praise be to Allah the Lord of the worlds and peace and blessings be upon our prophet Mohammed and his pure progeny

*I would like to express my thanks and deep gratitude to my supervisor **Dr. Hussein A.H. Al-Jboori** , and my supervisor **Dr.Nabaa N. Hasan** for them support and invaluable guidance through this work.and **Dr. Saheb Kehaid Jassim.***

*My special thanks to the head of the mathematics Department, College of Science, **Dr. Nadia Hashim AL-Noor**, and all staff members of this Department for their assistance and support during my study.*

I wish to express my deepest gratitude's to my family for their patience during my study.

Additionally, I would like to express my deep thanks to the head of the department of Mathematics and all staff members of this Department, College of Science, diyala University, for their encouragement, support and help, during my study.

Finally, I would like to thank all friends in life and those at the Department of Applied and Pure Mathematics.

Yasemeen M. Abd-Alhasan

September 2016

List of symbols

Symbol	Meaning
P_n	The space of polynomials of order n
$c(p)$	Positive constant depends on p
$J_n(f)$	Jackson polynomial
$L_{p,\alpha}(x)$	The space of all unbounded functions f on space X
$E_n(f)_p$	The degree of best approximation of functions on L_p(x)
$L_p(x)$	The space of all bounded measurable functions
T_M	The trigonometric polynomials of degree M
$\omega(f;\delta)$	The modulus of continuity of function f
$\omega_k(f;\delta)$	The modulus of smoothness of function f
$\Delta_h^k f(x)$	The kth symmetric difference of f
$\tau_k(f;\delta)_p$	The averaged modulus of smoothness on L_p(x)
$\omega_k(f,x;\delta)$	The local modulus of smoothness of function f
$C_{[a,b]}$	The space of all continuous bounded functions
$\tilde{E}_n(f)_p$	The best one-sided approximation of functions f on L_p(x)
$M_{[a,b]}$	The space of all bounded functions f on space X
π_r	The set of all algebraic polynomials of degree ≤r
$L_\infty(x)$	The space of all essentially bounded functions
$\varphi_n(x)$	Fejer kernels
$\hat{f}(k)$	The Fourier coefficients
$V_{2n,3n}(f,x)$	Valle-poussin polynomial
$c(p,k,l)$	Positive constant depending on p, k and l
L_n	Positive linear operators
$\partial_x^j f$	The partial derivatives of function f
τ_f^l	The averaged modulus of smoothness on L_p(x) with l derivatives
$\tau_f(\delta)$	The averaged modulus of smoothness

CONTENTS

ABSTRACT

This thesis is devoted to approximation of finite Fourier transform (FFT) that is (integrable, bonded and 1-periodic) function defined on [0,1] in L_p space , $1 \leq p < \infty$ and deal with what is called error estimated of function by positive linear operators via averaged modulus of smoothness.

The degree of best approximation for (FFT) in L_p −space , $1 \leq p < \infty$ by using trigonometric polynomials , Valle-poussin operator and L_n polynomials on [0,1] in terms of averaged modulus of smoothness have been studied and found.

The following results have been studied:

1. The error is estimate for (FFT) that is integrabel, 1-periodic and bounded function on [0,1] in L_p −space , $1 \leq p < \infty$ by trigonometric polynomials in terms averaged modulus of smoothness.

2. The error is estimate for (FFT) that is integrabel, 1-periodic and bounded function on [0,1] in L_p −space , $1 \leq p < \infty$ by Valle poussin operators in terms averaged modulus of smoothness.

3. The error is estimate for (FFT) that is integrabel, 1-periodic and bounded function on [0,1] in L_p −space , $1 \leq p < \infty$ by L_n polynomials in terms averaged modulus of smoothness.

Introduction

Our interest in approximation theory is related to its huge history as well as its plentiful connections with modern and classical analysis theory. The idea of approximation theory arose from methods to estimating the error in a large number of numerical processes such as interpolation and approximation of functions by means of operators.

Approximation theory of functions has been studied by a number of researchers, throughout trigonometric polynomials, algebraic polynomials, Jackson and Bernstein theorems . The Russian mathematical Chebyshev was the first person who invented the best approximation problem and how to change the linear motion into circular motion[14]. And also he developed in a very obvious way the goal of approximation theory in his researches.

Weierstrass developed a theorem that is considered the basic of approximation theory of real variable function and has great contribution in developing mathematical analysis which is for every continuous function f there exist an algebraic polynomials P_n of degree $\leq n$ such that the function f converges to P [31].

The methods of approximation are based on the use of new characteristic of functions, the name of new characteristics is averaged modulus of smoothness, or $\tau -$ moduli. They are defined for every bounded function integral, the averaged modulus of smoothness, which have already been used with success in a series of problems in the theory of approximation, are particularly helpful for estimating the error of functions that deal with finite numbers [37].

There are many researchers have studied the theory of approximation of functions as is given below :

In (1968), B. Sendov [8] introduced the ordinary L_p -modulus of continuity, denoted by $\omega - modulus.$

In (1984) Ivanov ,[20]"obtained some results about approximation of measurable and bounded functions by Bernstein polynomials in $L_p[0,1]$ space".

In (1987) , Z.Ditizian and V.Totik[10] introduced a way of measuring smoothness of functions ,where the need for these concepts arises from the failure of the classical moduli of smoothness to solve some basic problems.

In (1991) S. K. Jassim [22] presented an equivalence between the approximation of bounded measurable functions with trigonometric polynomial and the averaged modulus of smoothness in locally global space $(1 \leq p < \infty)$.

In (1999) , E. S. Bhaya [6] obtained some results about the convergence of periodic functions in the space $L_p(0 < p \leq 1)$,in terms of average modulus of smoothness .

R .Li and Y.Liu [33] (2008) , obtained the asymptotic estimations of best m-term approximation and Greedy algorithm for multiplier function classes defined by Fourier series .

In (2010) Leeka A.H [24], studied the approximation of unbounded functions by some algebraic and trigonometric polynomials in $L_{p,\alpha}-$ spaces and she found the degree of approximation of 2π −periodic unbounded functions by two trigonomiatric polynomials.

A.A. Hammod [17] (2012), introduced the estimation of any function (bounded and unbounded) by the k-functional and found estimation for positive linear operator by new weighted modulus of continuity in $L_{1,\alpha}-$ spaces .

The Scheme of this Thesis is:

In chapter one, modulus of continuity and average modulus of smoothness have been defined with properties , definitions , theorems and lemmas that need in our works are given.

Chapter two estimates the error for the finite Fourier transform in terms of trigonometric polynomials in L_p-norm via the averaged modulus of smoothness, and prove some results that related with it.

Chapter three estimates the error for the finite Fourier transform by averaged modulus of smoothness via Valle-poussin operator and proof some results that are related to it.

Chapter four estimates the error for the finite Fourier transform by averaged modulus of smoothness via L_n operator and proof some results that are related to it.

Chapter One

Fourier Series and Fourier Transform

1.1 Introduction

In this chapter, introduced many primary definitions theorems ,lemmas and corollaries, introduced Fourier series ,Fourier transform with applications and some of properties ,disecrete Fourier transform and fast Fourier transform.

1.2 Basic definitions and notations

In this section some definitions, theorems and lemmas that will be used in this work have been presented.

Definition (1.1)[40]

The real valued function f on $[a, b]$ is said to bounded if there is a real number M such that $|f(x)| \leq M$ for every $x \in [a, b]$ and otherwise it is said to be unbounded.

The following examples illustrate the bounded functions on the set of real number.

1) $f(x) = \sin x \qquad , x \in R, [-1,1]$

2) $f(x) = \cos x \qquad , x \in R, [-1,1]$

3) $f(x) = x \qquad , x \in [0,1]$

Definition (1.2)[12]

An algebra D of subset of some set X is called $\sigma - algebra$ if every union of a countable collection of members of D is again in D that is in addition to D being an algebra $\bigcup_{n=1}^{\infty} A_n$ belong to D for every$\{A_n\}$ sequence of D.

Definition (1.3)[12]

Let D be a $\sigma - algebra$ of subsets of X, a couple (X,D) is called a measurable space. A subset A of X is called measurable or measurable with respect to D if $A \in D$.

Definition (1.4)[40]

Let f a function defined on the measurable space X with values in the extended real numbers system the function f is said to be measurable if the set $\{x: f(x) > a\}$ measurable of every real a.

Definition (1.5)

Let $X = [0,1]$, then denoted by the L_p-space of all bounded measurable functions f on X such that:

$$\|f\|_{L_p} = \left\{\int_0^1 |f(x)|^p \, dx\right\}^{\frac{1}{p}} \quad , \qquad 1 \le p < \infty \tag{1.1}$$

Definition (1.6):[25]

Let $(X, \|\ \ \|)$ be a normed space , let Y be a non empty subspace of X and a point $x \in X$, if $y \in Y$ such that $\|x - y\| = inf_{z \in Y}\|x - z\|$ then y is called a best approximation from Y to X.

Definition (1.7)[38]

The degree of best approximation of $f \in L_p[0,1]$ with respect to the trigonometric polynomials of degree m in $L_p - spaces$ is defined by:

$$E_n(f)_p := inf_{p_n \in T_m}\|f - p_n\|_p \ , \qquad 1 \le p < \infty \tag{1.2}$$

Where T_m is the set of all trigonometric polynomials of degree m.

Theorem (1.8) (Existence best approximation) [29]

Let Y be finite dimensional subspace of X ,then for every $x \in X$ there exist best approximation form Y to X.

Definition (1.9)[12]

A set A is called convex if the set of points $\{\vartheta x_1 + (1-\vartheta)x_2\} \in A$ for every $x_1, x_2 \in A$, $0 < \vartheta < 1$ and A is called strictly convex if the points $\{\vartheta x_1 + (1-\vartheta)x_2\}$ are interior points for every $x_1, x_2 \in A$.

Definition (1.10)[12]

A normed spaces X is called strictly convex space if the ball B(0,1) is strictly convex.

Theorem (1.11) (uniqueness best approximation) [29]

In strictly convex normed space X there is at most one best approximation to an $x \in X$ out of a given subspace Y.

Definition (1.12)[38]

The modulus of continuity of f on [a ,b] is the following function of $\delta \in [0, b-a]$

$$\omega(f;\delta) = sup\{|f(x) - f(\acute{x})| : |x - \acute{x}| \leq \delta,\ x, \acute{x} \in [a,b]\} \qquad (1.3)$$

A necessary and sufficient condition for a function f to be continuous in the interval [a, b] is

$$\lim_{\delta \to 0} \omega(f;\delta) = \omega(f;0) = 0$$

Definition (1.13)[38]

The modulus of smoothness is a natural generalization of the modulus of continuity as the following:

$$\omega_k(f;\delta) = sup\{|\Delta_h^k f(x): |h| \le \delta, x, x+kh \in [a,b]|\} \quad (1.4)$$

k is positive integer $k=1,2,3,\ldots$

Where

$$\Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} f(x+mh)$$

Such that

$$\binom{k}{m} = \frac{k!}{m!\,(k-m)!}$$

is the binomial coefficient.

Example

when k=1 we have

$$\Delta_h^1 f(x) = f(x+2h) - f(x)$$

When $k=2$ we have

$$\Delta_h^2 f(x) = f(x) + f(x+2h) - 2f(x+h)$$

when $k=3$

$$\Delta_h^3 f(x) = 3f(x+h) + f(x+3h) - 3f(x+2h) \text{-} f(x)$$

The modulus of smoothness of order 1 is the modulus of continuity, i. e $\omega_1(f;\delta) = \omega(f;\delta)$

The modulus of smoothness of order 2, $\omega_2(f;\delta)$ called the Zigmund modulus

The modulus of smoothness have the following five basic properties :

1. Monotonicity:

$$\omega_k(f;\delta_1) \leq \omega_k(f;\delta_2) \qquad for\ 0 \leq \delta_1 \leq \delta_2$$

2. Semi-additivity:

$$\omega_k(f+g;\delta) \leq \omega_k(f;\delta) + \omega_k(g;\delta)$$

3. A higher order modulus can be estimated by means of a modulus of lower order

$$\omega_k(f;\delta) \leq 2\omega_{k-1}(f;\delta)$$

4. A modulus of the function can be estimated by means of a modulus of lower order modulus of the derivative:

$$\omega_k(f;\delta) \leq 2\omega_{k-1}(f\acute{;}\delta)$$

5. The integer multiplier for the modulus:

$$\omega_k(f;n\delta) \leq n^k\omega_k(f;\delta)$$

Where n is natural number

Definition (1.14)[38]

The averaged modulus of $f \in L_p[0,1\,]$, $1 \leq p < \infty$ with respect to the trigonometric polynomials in $L_p - spaces$ is given by:

$$\tau_k(f;\delta)_p = \|\omega_k(f,x;\delta)\|_{L_p} \qquad (1.5)$$

$$= \left[\int_a^b (\omega_k(f,x;\delta))^p dx \right]^{\frac{1}{p}}$$

Where

$$\omega_k(f,x;\delta) = \sup\left\{ |\Delta_h^k f(t)| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [0,1] \right\} \quad (1.6)$$

The function $\omega_k(f,x;\delta)$ is local modulus of smoothness.

Where

$$\omega_k(f;\delta) = \|\omega_k(f,.;\delta)\|_{C[0,1]}$$

The averaged modulus of smoothness have five basic properties:

1. Monotonicity:

$$\tau_k(f;\acute{\delta})_p \le \tau_k\left(f;\acute{\acute{\delta}}\right)_p \qquad for\ \acute{\delta} \le \acute{\acute{\delta}}$$

2. Semi-additivity:

$$\tau_k(f+g;\delta)_p \le \tau_k(f;\delta)_p + \tau_k(g;\delta)_p$$

3. A higher-order-modulus can be estimated by means of lower order:

$$\tau_k(f;\delta)_p \le 2\tau_{k-1}\left(f;\frac{k}{k-1}\delta\right)_p$$

4. The modulus of order k of the function can be estimated from the modulus of order k-1 of the derivative:

$$\tau_k(f;\delta)_p \leq \delta\tau_{k-1}\left(f;\frac{{}'k}{k-1}\delta\right)_p$$

5. The integer multiplier for the modulus:

$$\tau_k(f;n\delta)_p \leq (2n)^{k+1}\tau_k(f;\delta)_p$$

Where n is natural number.

Theorem(1.15)[38]

If f is a measurable bounded function on $[a\,,\,b]$ then

$$\omega_k(f;\delta)_p \leq \tau_k(f;\delta)_p \leq \omega_k(f;\delta)(b-a)^{\frac{1}{p}},\ 1 \leq p < \infty \qquad (1.7)$$

Weierstrass theorem (1.16)[15]

If $f \in C[a,b]$,then for each $\varepsilon > 0$,there exist trigonometric polynomial T such that

$$\|f - T\|_p < \varepsilon, \qquad (1 \leq p < \infty)$$

Definition (1.17)[42]

The value of a function repeats itself at regular intervals of x, called a periodic function.

A function f is periodic in x with periodic $n\pi$ if.

$f(x+n\pi) = f(x)$ $\quad n = 1,2,\ldots$ $\quad$ for all x.

For example, $y=\sin x$ is periodic in x with periodic π.

The following figure shows the form of periodic functions.

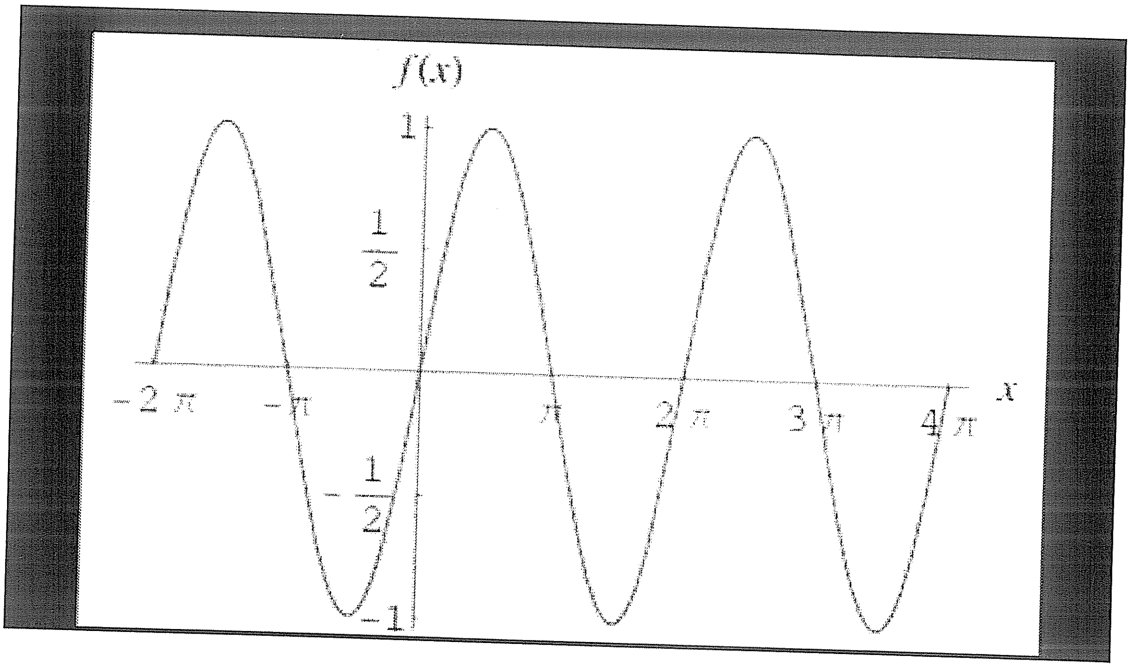

Figure (1.1) : periodic function

Theorem (1.18)[6]

Let f be a 2-periodic bounded measurable function $\frac{1}{2(l+1)} \leq p \leq 1$ then

$$E_n(f)_p \leq c(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_p ,k,l \in N \tag{1.8}$$

Theorem (1.19)[6]

Let $f \in L_p[a,b],\ \ a,b \in R,(0 < p \leq 1)$

$$\omega_k(f,\delta)_p \leq \tau_k(f,\delta)_p \tag{1.9}$$

Corollary (1.20)[7]

for every natural number k and $\delta > 0$,if $f \in M[a,b]$,then $\omega_k(f,x;\delta) \in L_p[a,b]$ as a function of x, $1 \leq p < \infty$.

Theorem (1.21)[5]

There exists a constant $c(k)$,depending only on $k \geq 2$,such that for every absolutely continuous function f on the interval $[a ,b]$

$$\tau_k(f;\delta)_p \leq c(k)\delta\omega_{k-1}(f';\delta)_p \tag{1.10}$$

Lemma (1.22)[18]

let $f \in L_p[-1,1]$,$1 \leq p < \infty$ $n \geq k$ then:

$$E_n(f)_p \leq C\omega_k(f,n^{-1})_p \ , \ \delta = n^{-1} \tag{1.11}$$

Where C is constant.

Lemma (1.23)[39]

Let $f \in L_p[-1,1]$ then

$$\tau_k(f,\delta)_p \leq C\|f\|_p \tag{1.12}$$

Where C is constant.

Lemma (1.24)[1]

Let $f \in L_p, (1 \leq p \leq \infty)$ then

$$\tau_k(f,\delta)_p \leq 2\tau_{k-1}(f',\delta)_p \tag{1.14}$$

Where $\delta > 0$

Theorem (1.25)[42]

Let $f \in L_p,\ 0 < p \leq \infty$ then there exists a polynomial

$Q_n \in \pi_r, (n \leq r)$ of degree $\leq n$,such that

$$\|f - Q_r\|_{L_p(X)} \leq C\omega_k(f, x; \delta)_p, \delta > 0 \tag{1.15}$$

Where C is constant.

1.3 Fourier Series

Fourier series is a way to represent a wave –like function as a combination of simple sine waves more formally, It decomposes any periodic function or periodic signal to the sum of a set of simple oscillating function namely sins and cosines.

The main idea behind Fourier theorem is: The function $\cos x$ and $\sin x$ each have period 2π, and in general the function $\cos nx$ and $\sin nx$ have period $2\pi/n$. But if the form of any linear combination of these functions that is ,multiply each by a constant and add the results to the resulting function still has period 2π . This lead us to the following:

Let *f(x)* be any periodic function with period 2π . The finite sum is

$$S_n(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) \tag{1.16}$$

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) \tag{1.17}$$

If $\lim_{n\to\infty} S_n = f(x)$

Then series (1.16) converges to *f(x)* and rewrite series (1.17)

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \tag{1.18}$$

Where the coefficients a_m and b_m are real numbers.

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx, \qquad m = 1,2,3,.... \tag{1.19}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx, \qquad m = 0,1,2,.... \tag{1.20}$$

And

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \tag{1.21}$$

If f is odd function-

$a_0 = 0$ and $a_m = 0$

$f(x) = \sum_{m=1}^{\infty}(b_m \sin mx)$

-If f is even function

$b_m = 0$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx)$$

1.4 Fourier Transform

There are several common conventions for defining the Fourier Transform as for an integral function (Kaiser 1994,pp. 29) [23]and (Rahman 2011,pp. 11) [32] this article use the following definition.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$$

for any real number ξ

When the independent variable x represents time the transform variable ξ

represents frequency under suitable conditions f is determined by via the inverse transform

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, dx$$

For any real number x

Fourier transform is a tool that breaks a wave form (a function or signal) in to an alternate representation characterized by sine and cosines.

Fourier transform shows that any wave form can be re-written as the sum of sinusoidal functions.

The Fourier transform a function of one variable time (in second)which lives in the time domain to second function which lives in the frequency domain (Hertz) and changes the basis of the function.

Frequency $\overset{Fourier\ transform}{\Longleftrightarrow}$ Time

The following figure shows the transform from time domain to frequency domain:

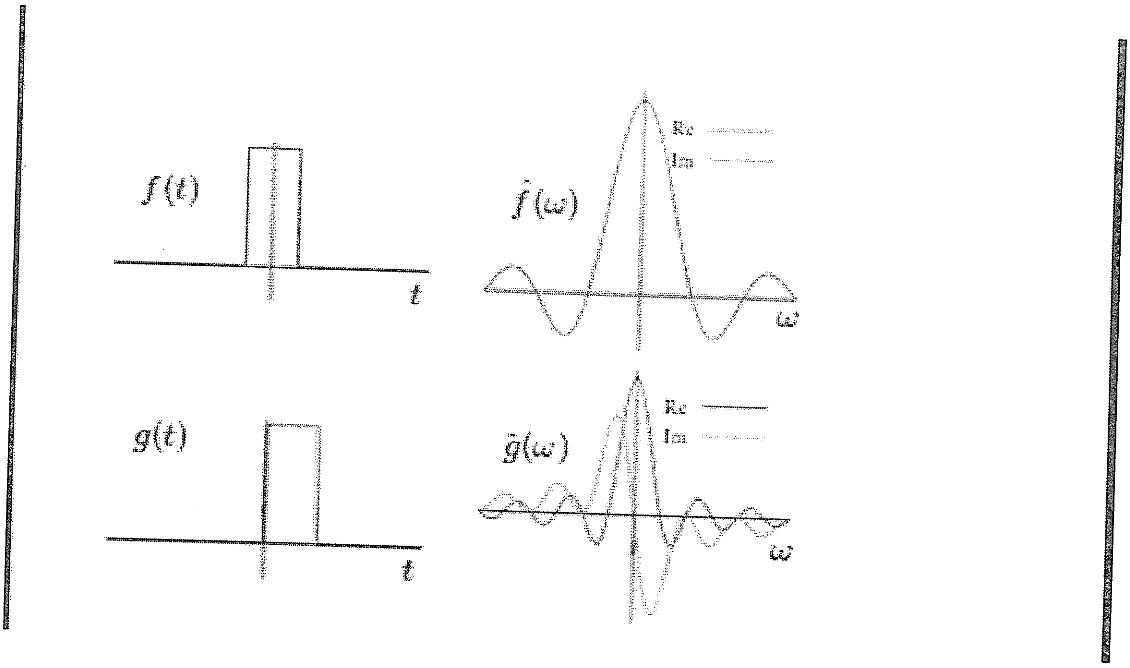

Figure (1.2) : Transform from time domain to frequency domain

$$F(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t} \quad dt$$

Its Fourier transform

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} \quad d\omega$$

It is inverse of Fourier transform

$F(t)$ represents time domain and $F(\omega)$ represents frequency domain.

Fourier transform is a mapping between the two domains

(Problem) $\xrightarrow{Fourier\ transform}$ (transform problem)

$solve \downarrow hard$ $\qquad\qquad$ $solve \downarrow easy$

(solution) $\xleftarrow{invers\ Fourier\ transform}$ (transform solution)

1.5 Applications of Fourier transform

There are many applications of Fourier series ,including any field of physical science that uses sinusoidal signals, such as engineering ,physics, applied mathematics and chemistry. Below are some important applications:

1. Analysis of Differential Equations

The most important use of the Fourier transform is to solve partial differential equations. Many of equations of the mathematical, physics can be treated this way. Fourier transform can be used to solve heat equation.

2. Fourier Transform Spectroscopy

The Fourier transform is used in nuclear magnetic resonance (NMR) and in other kinds of spectroscopy, e.g. infrared (FTIR). In NMR an exponentially shaped free induction decay (FID) signal is acquired in the time domain and Fourier-transform to the Lorentzian line-shape in the frequency domain. The Fourier transform is also used in magnetic resonance imaging (MRI) and mass spectrometry.

3. Quantum Mechanics

The Fourier transform is useful in Quantum mechanics in different ways. To begin with, the basic conceptual structure of Quantum mechanics postulates the existence of pairs of complementary variables, connected by the Heisenberg uncertainty principle.

4. Signal Processing

The Fourier transform is used for the spectral analysis of time-series. The subject of statistical signal processing dose not, however, usually apply the Fourier transform to the signal itself.

1.6 Some Properties of the Fourier transform [26]

1. Linearity

If α and β are any constant and created a new function $h(t) = \alpha f(t) + \beta g(t)$ as a linear combination of two old functions $f(t)$ and $g(t)$,then the Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} \quad dt = \int_{-\infty}^{\infty} [\alpha f(t) + \beta g(t)]\, e^{-i\omega t} dt$$

$$= \alpha \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \quad dt + \beta \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} \quad dt$$

$$= \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$$

2. Time Shifting

Suppose that created a new function $h(t) = f(t - t_0)$ by time shifting a function $f(t)$ by t_0.The easy way to check the direction of the shift is to note that if the original signal $f(t)$ has a jump when its argument $t=a$,then the new signal $h(t) = f(t - t_0)$ has a jump when $t - t_0 = a$,which is at $t=a+t_0$, t_0 units to the right of the original jump.

The Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} \quad dt = \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t} \quad dt$$

$$= \int_{-\infty}^{\infty} f(\grave{t}) e^{-i\omega(\grave{t}+t_0)} \; d\grave{t}$$

$$= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(\grave{t}) e^{-i\omega \grave{t}} \; d\grave{t}$$

Where

$$t = \grave{t} + t_0 \quad , \quad \mathrm{dt} = d\grave{t}$$

$$= e^{-i\omega t_0} \hat{f}(\omega)$$

3. Scaling

If created a new function $h(t) = f\left(\frac{t}{\alpha}\right)$ by scaling time by a factor of $\alpha > 0$,then the Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} \; dt = \int_{-\infty}^{\infty} f\left(\frac{t}{\alpha}\right) e^{-i\omega t} \; dt$$

Let $t = \alpha\grave{t}$, $dt = \alpha d\grave{t}$

$$= \alpha \int_{-\infty}^{\infty} f(\grave{t})\, e^{-i\omega\alpha\grave{t}} \; d\grave{t}$$

$$= \alpha \hat{f}(\alpha\omega)$$

4. Differentiation

If created a new function $h(t) = \grave{f}(t)$ by differentiating an old function $f(t)$,then the Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} \; dt = \int_{-\infty}^{\infty} \grave{f}(t)\, e^{-i\omega t} \; dt$$

Now integrate by parts with $u = e^{-i\omega t}$ and $dv = \grave{f}(t)dt$ so that $du = -i\omega e^{-i\omega t} \; dt$ and $v = f(t)$. suppose that $f(\mp\infty) = 0$,this gives

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} u \quad dv = uv|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} v \ du$$

$$\int_{-\infty}^{\infty} f(t)\,(-\mathrm{i}\omega)e^{-\mathrm{i}\omega t} \quad dt = i\omega\hat{f}(\omega)$$

5. Duality

The duality property tell us if created a new time-domain function $g(t) = \hat{f}(t)$by exchanging the roles of time and frequency , then the Fourier transform of g is.

$\hat{g}(\omega) = 2\pi f(-\omega)$

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} \ dt = \int_{-\infty}^{\infty} \hat{f}\,(t)e^{-i\omega t} \quad dt, \qquad s = t$$

$$= \int_{-\infty}^{\infty} \hat{f}(s)e^{-i\omega s} \quad ds$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} \quad d\omega\,, \qquad \omega = s$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{ist} \quad ds\,,$$

1.7 Discrete Fourier transform [21] [26]

Is one of the most important tools in digital signal processing ,the (Discrete Fourier Transform) can calculate a signals frequency spectrum . This is a direct examination of information encoded in the frequency ,phase, and amplitude of the component sinusoids. For example ,human speech and hearing use signals with this type of encoding . The DFT can find a systems frequency response from the systems impulse response ,and vice versa . This allows systems to be analyzed in the frequency domain ,just as convolution allows systems to be analyzed in the time domain . The DFFT can be used as an intermediate step in more elaborate signal processing techniques .

The sequence of N complex numbers $x_0, x_1, \dots, x_{N-1}$ is transform into an N-periodic sequence of complex numbers

$$X_k = \sum_{n=0}^{N-1} x_n . e^{-2i\pi kn/N} \qquad , k \in Z$$

And the inverse is

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k . e^{2i\pi kn/N} \qquad , n \in Z$$

There are many practical applications of (Discrete Fourier Transform) in [21],[26] :

In digital signal processing ,the function is any quantity or signal that varies over time ,such as the pressure of sound wave ,a radio or daily temperature readings ,sampled over a finite time interval . in image processing ,the samples can be the values of pixels along a row or column of a raster image . the (Discrete Fourier Transform) is also used to efficiently solve partial differential equations

The (Discrete Fourier Transform) formula can be converted to the trigonometric forms sometimes used in engineering and computer science:

Fourier transform:

$$X_k = \sum_{n=0}^{N-1} x_n \left(cos \left(-2\pi k \frac{n}{N} \right) + i \, sin \left(-2\pi k \frac{n}{N} \right) \right)$$

Inverse Fourier transform

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \left(cos\left(2\pi k\frac{n}{N}\right) + i\, sin\left(-2\pi k\frac{n}{N}\right)\right)$$

N is the number of time samples

N is the current sample (0,...N-1)

$x_n$is the value of the signal at time

K is the current frequency

$X_k$is the amount of frequency in the signal

1.8 Fast Fourier transform [37]

In 1985 a paper by J.W Cooley and J.W Tukey in the journal Mathematics of computation [CT] explain a new method of calculating the constants in the interpolating trigonometric polynomial. This method needing only $O(m\, log_2\, m)$ multiplications and $O(m\, log_2\, m)$ additions, provided m is chosen in an appropriate manner. For a problem with thousand of data points, this method reduces the number of calculations from millions to thousands.

This method is described by Cooley and Tukey and is known as Cooley-Tukey algorithm or the fast Fourier transform (FFT) algorithm and has led to revolution in the use of interpolating trigonometric polynomials .

The method consists of organizing the problem so that the number of data points being used can be easily factor , particularly into powers two.

Fast Fourier transform is an algorithm to compute the discrete Fourier transform (Discrete Fourier Transform) and its inverse quickly.

This quickly because the algorithm do not compute the parts that equal to zero in DFT.

Chapter Two

Error Estimate For The Finite Fourier Transform by Averaged Modulus of Smoothness

2.1 Introduction

Moduli of smoothness represent important tools in obtaining quantitative estimates of the error of approximation for positive processes[31].

In (2004) Epsten C.L. [9] estimated error for the continuous 1-periodic function (finite Fourier transform) (FFT) defined on closed interval in L_∞-spaces by modulus of continuity, now the error of the integrable ,bounded and 1-periodic functions on [0,1] by trigonometric polynomials in terms of averaged modulus of smoothness where $f \in L_p - spaces\,, 1 \leq p < \infty$ estimeted. This main theorem is proved.

$$\|f - p^*\|_p \leq 6\,\tau_f\left(\frac{1}{2\pi M}\right)$$

In (2000) A. H. Al-Abdulla [2] introduced new theorems concerning the convergence of periodic functions in the space $L_{p,\delta}(0 < p \leq 1; \delta > 0)$ in terms of averaged moduli of functions.

Trigonometric polynomials in the complex case are spanned by the positive and negative powers of e^{ix}.

Any function T of the form

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + i\sum_{n=1}^{N} b_n \sin(nx) \qquad (x \in R)$$

With a_n ,b_n in C for $0 \leq n \leq N$, is called a complex trigonometric polynomial of degree N (Rudin 1987, pp. 88). Using Eulers formula the polynomial can be rewriting as:

$$T(x) = \sum_{n=-N}^{N} c_n e^{inx} \qquad (x \in R)$$

Then

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx) \qquad (x \in R)$$

Is called a real trigonometric polynomial of degree N [28].

Also ,in this chapter introduced Whitney's theorem [38]

Considered f is a finite Fourier Transform functions.

2.2 Whitney theorem [38]

in 1957 Whitney prove the following theorem, which now classical in approximation theory and numerical analysis.

Let f be bounded function on [0,1] and integrable on [0,1], for integer $n \geq 1$, this notation important for prove.

$$h = \frac{1}{n+1}, \qquad x = vh + t, \qquad 0 \leq t \leq h$$

where v is integer.

The following operator is important to prove the theory :

$$\varphi_n(f; x) = \varphi_n(f; vh + t)$$

$$= \frac{(-1)^{n-v}}{h\binom{n}{v}} \int_0^h \Delta_y^n f(x - vy) dy$$

$$|\varphi_n(f; x)| \leq \frac{1}{\binom{n}{v}} \omega_n(f; h)$$

And by using the notation.

$$l_{n,v}(x) = \prod_{\substack{j=0 \\ j\neq v}}^{n} \frac{x-j}{v-j}, \qquad v = 0,1,\dots,n \tag{2.1}$$

Let us introduce some auxiliary results to prove main theorem:

Proposition (2.1) [38]

If f is an integrable on $[0,1]$,then for each integer $n \geq 1$

$$f(x) = P_{n-1}(x) + \varphi_n(f;x) + \sum_{j=0}^{n} \frac{1}{h}\int_0^t \varphi_n(f;jh+v) l'_{n,j}\left(\frac{x-v}{h}\right) dv \tag{2.2}$$

Where P_{n-1} is a polynomial of degree at most n-1.

Proposition (2.2) [38]

Let $P^*_{n-1}(f)$ be the interpolation polynomial for f at the point $h,2h,\dots,nh$ i.e

$$P^*_{n-1}(f;x) = \sum_{j=1}^{n} f(ih) l_{n-1,j-1}\left(\frac{x}{h}-1\right) \tag{2.3}$$

Then

$$f(x) - P^*_{n-1}(f;x) =$$
$$\Delta_h^n f(0) l_{n,0}\left(\frac{x}{h}\right) + \varphi_n(f;x) - \sum_{j=0}^{n} \varphi_n(f;jh) l_{n,j}\left(\frac{x}{h}\right) +$$
$$h^{-1}\int_o^t \sum_{j=o}^{n} \varphi_n(f;jh+v)\, l'_{n.j}\left(\frac{x-v}{h}\right) dv \tag{2.4}$$

Lemma(2.3) [38]

Let

$$y_n := max\left\{\sum_{j=1}^{n} \binom{n}{j}^{-1} |l_{n,j}(x)| : 0 \leq x \leq 1\right\} = 1 \tag{2.5}$$

Lemma(2.4) [38]

Let

$$\mu_{n,v} := \sum_{j=o}^{n} \binom{n}{j}^{-1} max\{|l_{n,j}(x)|: v \leq x \leq v+1\} \quad (2.6)$$

Then

$$\mu_{n,v} \leq \frac{1+\sigma_v+\sigma_{v+1}}{\binom{n}{v}} \quad (2.7)$$

$v = 0,1,2,\ldots\frac{n-1}{2}$

Where

$$\sigma_v := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{v} \quad ,\sigma_0 := 0$$

Inparticular ,for v=0, then

$$\mu_{n,o} = 1 + \sum_{j=1}^{n} \binom{n}{j}^{-1} max\{|l_{n,j}(x)|: o \leq x \leq 1\} \leq 2 \quad (2.8)$$

Proposition (2.5) [38]

For any function f which is integrable on $[0,1]$, then

$$\left| f(x) - \sum_{j=1}^{n} f(jh) l_{n-1,j-1}\left(\frac{x}{h} - 1\right) \right|$$

$$\leq \frac{6 + 7min(\sigma_v, \sigma_{n-v})}{\binom{n}{v}} \omega_n(f;h) \quad (2.9)$$

For $x \in [vh, (v+1)/h$, h:=1/(n+1) , v = 0,1,…,n , $\sigma_v := 1 + \frac{1}{2} + \cdots + \frac{1}{v}$

$$\sigma_0 := 0$$

Theorem(2.6)[38]

For any function f integrable on [0,1] and for each $n \geq 1$ there is a polynomial p of degree at most n-1 such that .

$$|f(x) - p(x)| \leq 6\omega_n\left(f; \frac{1}{n+1}\right) \tag{2.10}$$

2.3 Some important notations [9]

First, defined a set of complex sequences of length $N+1$,

$$S_N = \{\langle c_0, c_1, \dots, c_N\rangle : c_j \in C, \quad j = 0, \dots, N\}$$

the Finite Fourier Transform *(FFT)* is the map from S_N to itself such that

$$F_{N(\langle c_j\rangle)_k} = \frac{1}{N+1}\sum_{j=0}^{N} c_j\, e^{\frac{-2\pi ijk}{N+1}} \tag{2.11}$$

This sequence is periodic, it can be extended to all $K \in Z$, it is natural to think

$k_{min=-\left[\frac{N+1}{2}\right]}$ to $k_{max=\left[\frac{N}{2}\right]}$

The inverse of *(FFT)* is given as

$$F_N^{-1}(\langle c_j\rangle) = \sum_{k=min}^{kmax} c_j\, e^{\frac{2\pi ijk}{N+1}} \tag{2.12}$$

Aprincipal application of the *(FFT)* is to approximately compute samples of the Fourier transform of a function. If f is a function defined on [0,1] then we define

$$\tilde{f}_{N,k} = \frac{1}{N+1}\sum_{j=1}^{N} f\left(\frac{j}{N+1}\right) e^{\frac{-2\pi ijk}{N+1}} \tag{2.13}$$

The sum on the right-hand side of equation (2.13) is Riemann sum

$$\hat{f}(k) = \int_o^1 f(x)e^{-2\pi ikx}\,dx \tag{2.14}$$

A trigonometric polynomials is a functions of the form

$$P(x) = \sum_{-M}^{M} \alpha_k\, e^{2\pi ikx} \qquad \alpha_k \in C \tag{2.15}$$

We denote the set of such functions by T_M ,if f is (FFT) ,an integrable,1-periodic function defined on [0,1] ,then for each $M \in N$,there is a function $p^* \in T_M$ that is a best approximation to f such that

$$\|f - p^*\|_p = \inf_{p \in T_M} \|f - p\|_p \tag{2.16}$$

2.4 The error estimate to best approximation for (FFT)

Introduced the main results about estimating the error for the finite Fourier transform by averaged modulus of smoothness by using trigonometric polynomial T_M.

Theorem (2.7)

let $f(x)$ be (Finite Fourier Transform) is an integrable, 1-periodic function on [0,1] then the best approximation p^* to f in T_M satisfies

$$\|f - p^*\|_p \le 6\,\tau_f\left(\frac{1}{2\pi M}\right) \tag{2.17}$$

Proof

Since the interpolation polynomial

$$P_{n-1}^*(f;x) = \sum_{j=1}^{n} f(jh) l_{n-1,j-1}\left(\frac{x}{h} - 1\right)$$

$$\|f(x) - p_{n-1}^*(f,x)\|_p$$

$$\leq \left\| \Delta_h^n f(0) l_{n,0}\left(\frac{x}{h}\right) + \varphi_n(f;x) \right.$$

$$- \sum_{j=o}^{n} \varphi_n(f;jh)\left(\frac{x}{h}\right)$$

$$\left. + h^{-1} \int_0^t \sum_{j=0}^{n} \varphi_n(f;jh+v) l'_{n,j}\left(\frac{x-v}{h}\right) dv \right\|_p$$

Using

$$\varphi_n(f;x) = \varphi_n(f;vh+t)$$

$$= \frac{(-1)^{n-v}}{h\binom{n}{v}} \int_0^h \Delta_y^n f(x-vy)dy$$

$$\|\varphi_n(f;x)\|_p \leq \frac{1}{\binom{n}{v}} \tau_n(f;h)$$

$$\|f(x) - P^*_{n-1}(f;x)\|_p$$

$$\leq \left\| \left[\tau_n(f;h) \max_{0\leq t\leq 1} |l_{n,0}(t)| + \tau_n(f;h) \right.\right.$$

$$+ \tau_n(f;h) \max_{0\leq t\leq 1} \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} |l_{n,j}(t)|$$

$$\left.\left. + \tau_n(f;h) \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} \int_0^t |l'_{n.j}(v)|\, dv \right] \right\|_p$$

$$= \left\| \tau_n(f;h) \left[max_{0\leq t\leq 1} |l_{n,0}(t)| + 1 + max_{0\leq t\leq 1} \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} |l_{n,j}(t)| + \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} \int_0^t |l'_{n.j}(v)|\, dv \right] \right\|_p$$

$$\leq \tau_n(f;h) \left[3 + 1 + \sum_{j=1}^{n} \frac{1}{\binom{n}{j}} \max_{0\leq t\leq 1} |l_{n,j}(t)| \right]$$

$$\leq \tau_n(f;h) \left[4 + \frac{1 + \sigma_0 + \sigma_1}{\binom{n}{o}} \right]$$

Then

$$\|f - p^*\|_p \leq 6\, \tau_f \left(\frac{1}{2\pi M} \right) \quad ■$$

The following corollary for the theorem (2.7)

Corollary (2.8)

If f is an integrable,1-periodic function on [0,1] with l integrable 1-periodic dervatives ,then the best approximation $p*$ to f satisfies

$$\|f - p^*\|_p \leq 6^{l+1} \frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l} \tag{2.18}$$

The following results proved.

Theorem (2.9)

For $M \in N$ and f, an integrable 1-periodic function defined on [0,1] we have estimates

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|p^* - f\|_p \qquad for\ |k| \leq M \tag{2.19}$$

Where p* ∈ T_M

Proof.

Since $p^* \in T_M$, for $|k| \leq M$ we have

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| = \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k} + \tilde{p}^*_{2M,k} - \hat{f}_{(k)}\right|$$

Then

$$\left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right|$$

$$= \left\|\left\| \int_0^1 \left(\frac{1}{2M+1} \sum_{j=0}^{2M} \left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right] e^{\frac{-2\pi ijk}{2M+1}} \right)^p dx \right\|\right\|^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left(\left|\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right] e^{\frac{-2\pi ijk}{2M+1}}\right|\right)^p dx\right]^{\frac{1}{p}}$$

$$\left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| \leq \|f - p^*\|_p$$

$$\left|\tilde{p}^*_{2M,k} - \hat{f}_{(k)}\right| = \left\|\left[\int_0^1 \left(\int_0^1 [f(x) - p^*(x)]\, e^{-2\pi ikx}\right)^p dx\right]^{\frac{1}{p}}\right\|$$

$$\leq \left[\int_0^1 \left|\left(\int_0^1 [f(x) - p^*(x)] e^{-2\pi ikx}\right)^p\right| dx\right]^{\frac{1}{p}}$$

$$\left|\hat{f}(k) - \tilde{p}_{2M,k}\right| \leq \|f - p^*\|_p$$

Then we use the triangle inquality.

$$\left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k} + \hat{p}^*_{2M,k} - \hat{f}_{(k)}\right| \leq \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| + \left|\hat{f}_{(k)} - \hat{p}^*_{2M,k}\right|$$

$$\left|\tilde{f}_{2M,k} - \hat{f}\ (k)\right| \leq \|f - p^*\|_p + \|f - p^*\|_p$$

$$\leq 2\|p^* - f\|_p \qquad \blacksquare$$

Corollary (2.10)

Suppose that f is an integrabal 1-periodic function with $l \geq 0$ an integrabal 1-periodic derivatives ; then

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 12\frac{6^l \tau_f^l\left(\frac{1}{2M\pi}\right)}{(2\pi M)^l} \qquad (2.20)$$

For $|k| \leq M$

Proof .

From theorem (2.9)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|p^* - f\|_p$$

Then from corollary (2.8)

$$\|f - p^*\|_P \leq 6^{l+1} \frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Then
$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2.\,6^{l+1} \frac{\tau^l_f\left(\frac{1}{2M\pi}\right)}{(2M\pi)^l}$$

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 12 \frac{6^l \tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$ ■

Corollary (2.11)

Suppose that f is an integrable 1-periodic function then

$$\left|\tilde{f}_{2M,k}\right| \leq \left|\hat{f}(k)\right| + 6\tau_f\left(\frac{1}{2\pi M}\right) \tag{2.21}$$

Proof .

By theorem (2.9)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|f - p^*\|_p$$

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k} + \tilde{p}^*_{2M,k} - \hat{f}(k)\right|$$

$$\leq \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| + \left|\tilde{p}^*_{2M,k} - \hat{f}(k)\right|$$

$$\left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| \leq \|f - p^*\|_p$$

Then from theorem (2.7)

$$\|f - p^*\|_p \le 6\,\tau_f\left(\frac{1}{2\pi M}\right)$$

Then

$$|\tilde{f}_{2M,k}| \le \|f - p^*\|_p + |\hat{f}(k)|$$

$|\tilde{f}_{2M,k}| \le 6\,\tau_f\left(\frac{1}{2\pi M}\right) + |\hat{f}(k)|$ ■

Theorem (2.12)

If f is an integrabal 1-periodic function ,then There is a constant C such that

$$|\tilde{f}_{2M(x)} - f_{2M(x)}| \le C \log M\,\|f - p^*\|_p \qquad (2.22)$$

Where $p^* \in T_M$ is a best a pproximation .

Proof :

Let Dirichlet kernel

$$f_{2M}(x)=\int_0^1 D_M\,(x - y)f(y)dy$$

Then

$$\tilde{f}_{2M}(x) = \frac{1}{2M+1}\sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right)f\left(\frac{j}{2M+1}\right)$$

We observe that

$$|\tilde{f}_{2M(x)} - f_{2M(x)}| \le |\tilde{f}_{2M(x)} - p^*(x)| + |p^*(x) - f_{2M(x)}|$$

$$\leq \left[\left\| \int_0^1 \left(\frac{1}{2M+1} \sum_{j=0}^{2M} D_M \left(x - \frac{j}{2M+1} \right) \left[f\left(\frac{j}{2M+1} \right) - p^*\left(\frac{j}{2M+1} \right) \right] \right.\right.\right.$$

$$\left.\left.\left. + \int_0^1 D_M (x-y)[p^*(y) - f(y)]dy \right)^p dx \right\|\right]^{\frac{1}{P}}$$

$$\leq \left[\int_0^1 \left| \left(\frac{1}{2M+1} \sum_{j=0}^{2M} D_M \left(x - \frac{j}{2M+1} \right) \left[f\left(\frac{j}{2M+1} \right) - p^*\left(\frac{j}{2M+1} \right) \right] \right.\right.\right.$$

$$\left.\left.\left. + \int_0^1 D_M (x-y)[p^*(y) - f(y)]dy \right)^p \right| dx \right]^{\frac{1}{P}}$$

$$\leq \|f - p^*\|_p \cdot \left[\frac{1}{2M+1} \sum_{j=0}^{2M} \left| D_M \left(x - \frac{j}{2M+1} \right) \right| + \int_0^1 |D_M(x-y)|\, dy \right]$$

Both the sum and the integral in the last line are bounded by a constant times $\log M$

■

Corollary (2.13)

If f is an integrabal 1-periodic function whose averaged modulus satisfies

$$\tau_f(\delta) = o(|\log \delta|^{-1}) \tag{2.23}$$

Where o is Landow symbol defined as:

$$f \in o(g) = \lim_{x \to 0} \left| \frac{f(x)}{g(x)} \right| = 0$$

Then the (FFT) partial sum $(\tilde{f}_{2M})$ converges to f on $[0,1]$ if f has l integrabal periodic derivatives and τ_f^l satisfies the (2.23), then for each $1 \leq j \leq l$,the sequence$\langle \partial_x^i \tilde{f}_{2M} \rangle$ converges to ∂_x^jf

Proof

Let p*∈ $T_M$be a best approximation to f ,we will use the triangle inquality to conclude that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq \left|\tilde{f}_{2M(x)} - p^*(x)\right| + \left|p^*(x) - f_{2M(x)}\right|$$

Applying theorem (2.12)and (2.7)

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq C \log M \, \|f - p^*\|_p$$

Since

$$\|f - p^*\|_p \leq 6\, \tau_f \left(\frac{1}{2\pi M}\right)$$

Then

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq (C \log M + 6)\tau_f \left(\frac{1}{2M\pi}\right) \quad (2.24)$$

The estimate in (2.23) implies the right-hand side of equation (2.24) tends to zero as M tends to infinity.

■

Chapter Three

Error Estimate For The Finite Fourier Transform via Vall Poussin Operators

3.1 Introduction

In (2004) Epsten C.L. [9] estimate error for the continuous 1-periodic function (finite Fourier transform) (FFT) defined on closed interval in L_∞-spaces by modulus of continuity, in this chapter we will state and prove estimate the error of the Finite Fourier Transform via Valle-Poisson operator by averaged modulus of smoothness, for 1-periodic, integrale function defined on [0,1],in L_p-spaces and state results that relation with it .

Vallee-poussin is a French mathematical whose first mathematical research was on analysis, in particular concentrating on integrals and solution of differential equations. Also he worked on approximation to functions by algebraic and trigonometric polynomials.

In (2010) Lekaa A.H. [24] used Valle-poisson operator to find out a unique trigonometric polynomial of degree (2n) interpolating 2π −periodic unbounded functions f at the points $\{X_j\}_{j=0}^{3n}$, is denoted by $V_{2n,3n}(f,x)$ and prove a direct inequality of 2π −periodic unbounded functions in $L_{p,\alpha}$–spaces $\left(\frac{1}{2(l+1)} \leq p \leq 1\right)$, $l = 1,2,\ldots$ via averaged modulus of smoothness.

In (2014) Alaa A.A. [1] introduced the estimate the degree of best one –sided approximation of unbounded function in $L_{p,w}(X)$, by Valle-Poisson operator in terms of averaged modulus of smoothness.

3.2 Definitions and Notations:

Valle-Poisson is positive linear operator [1].

A polynomials operator is a function of the form:

$$V_{n,m}(u) = \frac{1}{(m-n+1)}\{D_n(u) + D_{n+1}(u) + \cdots + D_m(u)\}$$

$$V_{n,m}(u) = D_n(u)$$

$$+ (m+1)\Big/(m-n+1) \sum_{k=n+1}^{m} [1 - k/(m+1)] \cos ku$$

$$V_{n,m}(u) = \frac{1}{(m-n+1)}\{(m+1)k_m(u) - nk_{n-1}(u)\}$$

$$= \frac{\sin\frac{m+n+1}{2}u. \qquad \sin\frac{m-n+1}{2}\ u}{2(m-n+1)sin^2\frac{u}{2}}.$$

Where $D_n(u)$ is the Dirichlet kernel of degree n.

$$D_n(u) = \frac{1}{2}\sum_{k=1}^{n} \cos ku = [\sin(n+1/2)u]/[2\sin(u/2)]$$

$m > 2n$ *and* $u \in [-\pi, \pi]$

$$V_{2n,m}(f,x) = \frac{2}{m+1}\sum_{k=0}^{m} f(x_{k,m})\, V_{n,2n}(x - x_{k,m})$$

Where

$$x_{k,m} = \frac{2\pi k}{m+1} \qquad ,k = 0,1,\ldots..m$$

Be the Jackson polynomial of function f and.

$$V_{n,m}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(u)\,.\,V_{n,m}(u-x)\ du \qquad (3.1)$$

If f is an integrable ,1-periodic function defined on [0,1] then for each $m \in N$, there is a polynomials operator $V_{n,m}$ of best approximation of f such that

$$E_n(f) \leq \|f - V\_(n,m)\ (f,x)\ \|_p$$

Where $E_n(f)$ is the degree of best approximation of functions.

Theorem (3.1) (de la Valle-Poisson) [15]

For each positive integer n, there exists a polynomial operator V_n of degree $2n-1$ with property .

$$\|f - V_n(f)\| \leq 4E_n(f)$$

We have some properties of the operators V_n

1. V_n is a trigonomatric polynomial operator of degree 2n-1, and $V_n(T_n) = T_n$.
2. we have $V_n(f) \to (f)$ for $n \to \infty$
3. $\|V_n\| \leq 3$ Indeed, $\|V_n\ \|_p \leq 2\|\sigma_{2n}\| + \|\sigma_n\| = 3$

Lemma (3.2)[24]

Let f, g be two functions defined on the same domain, then

(i) $V_{n,m}(f+g) = V_{n,m}(f) + V_{n,m}(g)$

(ii) $V_{n,m}(\alpha f) = \alpha V_{n,m}(f)$

Where α is constant.

Lemma (3.3) [24]

If $f \in 2\pi$ −periodic bounded measurable function, then

$$\|f - V_{2n,3m}(f)\|_p \le c(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_p$$

Where

$n = 1,2,\ldots$ $\qquad$ p, k and l are constant depends on p.

3.3 Error estimate for (FFT) via Valle-Poussin operator

The main theorem about estimating the error of the finite Fourier transform via Valle-poussin by averaged modulus of smoothness proved.

Theorem (3.4)

If *f(x)* is *FFT* an integrable ,1-periodic function on [0,1],then the polynomial operator of best approximation satisfies

$$\|f - V_{n,m}\|_p \le 4\tau_f\left(\frac{1}{n+1}\right)$$

Proof

$$\|f - V_{n,m}(f,x)\|_p = \|f - P_n - V_{n,m}(f - P_n)\|_p$$

Where P_n is a polynomial of order n of the form

$$P(x) = \sum_{-M}^{M} \alpha_k\, e^{2\pi kx} \qquad \alpha_k \epsilon C$$

$$\le \|f - P_n\|_p + \|V_{n,m}(f - P_n)\|_p$$

$$\le \|f - P_n\|_p(1 + \|V_{n,m}\|)$$

Since $\quad \|V_{n,m}\| \le 3$

$$\|f - V_{n,m}\|_p \le 4\|f - P_n\|_p$$

$$= 4E_n(f)_p$$

From theorem (1.18),we get

$$\|f - V_{n,m}\|_p \leq 4\tau_f\left(\frac{1}{n+1}\right) \qquad \blacksquare$$

The following corollary to (3.4) theorem.

Corollary (3.5)

If f is an integrable 1-periodic function on [0,1] with l an integrable 1-periodic derivatives then the polynomial operator of best approximation satisfies

$$\|f - V_{n,m}\| \leq 4^{l+1}\tau_f\left(\frac{\frac{1}{n+1}}{(n+1)^l}\right)$$

Theorem (3.6)

For $M \in N$and f ,an integrable 1-periodic function defined on [0,1],we have

$$|\tilde{f}_{2M,k} - \hat{f}_{(k)}| \leq 2\|V_{n,m} - f\|_p |k| \leq M \qquad , M \geq 2n$$

Proof :

Since $V_{n,M}$ is polynomial operator and $|k| \leq M$

$$|\tilde{f}_{2M,k} - V_{n,M} + V_{n,M} - \hat{f}_{(k)}|$$

Then

$$\leq |\tilde{f}_{2M,k} - V_{n,M}| + |V_{n,M} - \hat{f}_{(k)}|$$

$$\left|\tilde{f}_{2M,k} - V_{n,M}\right| = \left[\left|\int_0^1 \left(\frac{1}{2M+1}\sum_{j=0}^{2M} f\left(\frac{j}{2M+1}\right) e^{\frac{-2\pi jk}{2M+1}} - \frac{2}{m+1}\sum_{k=0}^{m} f(x_{k,m})\, V_{n,2n}(x - x_{k,m})\right)^p\right|\right]^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left|\left(\frac{1}{2M+1}\sum_{j=0}^{2M} f\left(\frac{j}{2M+1}\right) e^{\frac{-2\pi jk}{2M+1}} - \frac{2}{m+1}\sum_{k=0}^{m} f(x_{k,m})\, V_{n,2n}(x - x_{k,m})\right)^p\right| dx\right]^{\frac{1}{p}} \left|\tilde{f}_{2M,k} + V_{n,M}\right| \leq \left\|V_{n,M} - f\right\|_p$$

And

$$\left|V_{n,M} - \hat{f}_{(k)}\right| = \left[\left|\int_0^1 \left(\frac{1}{\pi}\int_0^{2\pi} f(u)\,.\,V_{n,m}(u-x)\ du - \int_0^1 f(x)e^{-2\pi kx}\, dx\right)^p dx\right|\right]^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left(\left|\frac{1}{\pi}\int_0^{2\pi} f(u)\,.\,V_{n,m}(u-x)\ du\ - \int_0^1 f(x)e^{-2\pi kx}\, dx\right|\right)^p dx\right]^{\frac{1}{p}}$$

$$\left|V_{n,M} - \hat{f}_{(k)}\right| \leq \left\|V_{n,M} - f\right\|_p$$

So that

$$\left|\tilde{f}_{2M,k} - \hat{f}_{(k)}\right| \leq 2\left\|V_{n,M} - f\right\|_p \qquad \blacksquare$$

Corollary (3.7)

Suppose that f is an integrable 1-periodic function with l an integrabal 1-periodic derivatives; then

$$\left|\tilde{f}_{2M,k} - \hat{f}_{(k)}\right| \leq 8\frac{4^l \tau_f^l\left(\frac{1}{n+1}\right)}{(n+1)^l}$$

Proof :

From theorem(3.6)

$$\left|\tilde{f}_{2M,k} - \hat{f}_{(k)}\right| \leq 2\left\|V_{n,M} - f\right\|_p$$

And corollary (3.5)

$$\left\|f - V_{n,m}\right\| \leq 4^{l+1}\tau_f\left(\frac{\frac{1}{n+1}}{(n+1)^l}\right)$$

Then

$$\left\|f - V_{n,m}\right\| \leq 2.4^{l+1}\tau_f\left(\frac{\frac{1}{n+1}}{(n+1)^l}\right)$$

$$\left|\tilde{f}_{2M,k} - \hat{f}_{(k)}\right| \leq 8\, 4^l\tau_f\left(\frac{\frac{1}{n+1}}{(n+1)^l}\right)$$ ∎

Corollary (3.8)

Suppose that f is an integrable 1-periodic function then

$$\left|\tilde{f}_{2M,k}\right| \leq \left|\hat{f}_{(k)}\right| + 4\tau\left(f, \frac{1}{n+1}\right)$$

Proof:

From theorem (3.6)

$$\left|\tilde{f}_{2M,k} - \hat{f}_{(k)}\right| \leq 2\left\|V_{n,M} - f\right\|_p$$

And

$$\left|\tilde{f}_{2M,k} + V_{n,M}\right| \leq \left\|V_{n,M} - f\right\|_p$$

Then from theorem (3.4)

$$\left\|f - V_{n,m}\right\| \leq 4\tau_f\left(\frac{1}{n+1}\right)$$

$$\left|\tilde{f}_{2M,k}\right| \leq \left\|V_{n,M} - f\right\|_p + \left|\hat{f}_{(k)}\right|$$

$$\left|\tilde{f}_{2M,k}\right| \leq \left|\hat{f}_{(k)}\right| + 4\tau\left(f, \frac{1}{n+1}\right)$$ ■

Theorem(3.9)

There is a universal constant C such that if f is an integrable ,1-periodic function then

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq C \log M \left\|V_{n,M} - f\right\|_p$$

Proof:

Let Dirichlet kernel

$$f_{2M(x)} = \int_0^1 D_M(x-y) f_{(y)}\, dy$$

Then

$$\tilde{f}_{2M,k} = \frac{1}{2M+1}\sum_{j=o}^{2M} D_M\left(x - \frac{j}{2M+1}\right) f\left(\frac{j}{2M+1}\right)$$

And

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \left|\leq \left|\tilde{f}_{2M(x)} + V_{n,M} - V_{n,M} - f_{2M(x)}\right|\right|$$

$$\leq \left|\tilde{f}_{2M(x)} - V_{n,M}\right| + \left|V_{n,M} - f_{2M(x)}\right|$$

$$\leq \left[\left\| \int_0^1 \left(\frac{1}{2M+1}\sum_{j=o}^{2M} D_M\left(x - \frac{j}{2M+1}\right) f\left(\frac{j}{2M+1}\right) - \frac{2}{m+1}\sum_{k=0}^{m} f(x_{k,m})\, V_{n,2n}(x - x_{k,m}) + \int_0^1 D_M(x-y)[V_{(y)} - f_{(y)}] \right)^p \right\|\right]^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left|\left(\left(\frac{1}{2M+1}\sum_{j=o}^{2M} D_M\left(x - \frac{j}{2M+1}\right) f\left(\frac{j}{2M+1}\right) - \frac{2}{m+1}\sum_{k=0}^{m} f(x_{k,m})\, V_{n,2n}(x - x_{k,m}) + \int_0^1 D_M(x-y)[V_{(y)} - f_{(y)}]\, dy \right)\right)^p \right| dx\right]^{\frac{1}{p}}$$

$$\left\|V_{n,M} - f\right\|_p \cdot \frac{1}{2M+1}\sum_{j=o}^{2M} \left|D_M\left(x - \frac{j}{2M+1}\right)\right| + \int_0^1 D_M |(x-y)| dy$$

These eq. Bounded by a constant log M ∎

Corollary (3.10)

If f is an integrable 1-periodic function whose averaged modulus satisfies

$$\tau_f(\delta) = o\left(|\log \delta|^{-1}\right)$$

The FFT partial sum ($\tilde{f}_{2M}$) convergs to f on [0,1] if f has l integrabal periodic function derivatives and τ_f^l satisfies the above estimate then for each $1 \leq j \leq l$ the sequence $\langle \partial_x^j \tilde{f}_{2M} \rangle$ converges to $\partial_x^j f$

Proof :

Let p^*be a best approximation to f ,we will use the triangle inequality to concluded that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \left|\leq \left|\tilde{f}_{2M(x)} + V_{n,M} + V_{n,M} - f_{2M(x)}\right|\right|$$

$$\leq \left|\tilde{f}_{2M(x)} - V_{n,M}\right| + \left|V_{n,M} - f_{2M(x)}\right|$$

Apply theorem (3.9)

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq C \log M \ \left\|V_{n,M} - f\right\|_p$$

Since

$$\left\|f - V_{n,m}\right\| \leq 4\tau_f\left(\frac{1}{n+1}\right)$$

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq (C \log M \ + 4)\tau_f\left(\frac{1}{n+1}\right)$$

$\left\|\tilde{f}_{2M(x)} - f_{2M(x)}\right\|$ tends to zero as M tends to infinity. ■

Chapter Four

Error Estimate For The Finite Fourier Transform in Terms of L_n polynomial

4.1 Introduction

In (2004) Epsten C.L. [9] estimated error for the continuous 1-periodic function (finite Fourier transform) (FFT) defined on closed interval in L_∞-spaces by modulus of continuity, in this chapter estimated the error for the finite Fourier transform by averaged modulus of smoothness via L_n polynomial in L_p-spaces on [0,1] for 1- periodic ,integrabal functions and prove the main theorem

$$\|f - L_n\|_p \leq 6\, \tau_k\left(f, \sqrt{x(1-x)/n}\right)$$

4.2 Definitions and notations [11]

Let $J_n(t)$ denote the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2}\right)^8 \quad , \quad \int_{-\pi}^{\pi} J_n(t)dt = 1$$

And define

$$T_j(t) = \int_{t-t_j}^{t+t_j} J_n(u)du \quad , j = 0,1, \dots \dots, n$$

Where $t_j = j\pi/n$, $j = 0, \dots, n$. now for $x = \cos t$ let $r_j(x) = T_{n-j}(t)$ and define

$$R_j(x) = \int_{-1}^{x} r_j(u)du, \qquad j = 0, \dots, n$$

Not that since $T_0 \equiv 0$ and $T_n \equiv 1$ we have $R_0(x) = 1 + x$ and $R_n(x) \equiv 0$.

The point ξ_j are defined by the equations $1 - \xi_j = R_j(1)$. since

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \qquad j = 0, \dots, n-1$$

And if $\varphi_j = (x - \xi_j)_+$ we can write

$$S_n(x) = f(-1) + s_0(1 + x) + \sum_{j=1}^{n-1} (s_j - s_{j-1})\varphi_j(x)$$

S_n is linear interpolant in $[\xi_{j+1} - \xi_j]$,replacing $\varphi_j(x)$ by $R_j(x)$

Then $R_j(x)$ brings us to the polynomials

$$L_n(f) = f(-1) + s_0 R_0 + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j$$

$$= f(-1) + \sum_{j=0}^{n-1} s_j \left(R_j - R_{j+1}\right)$$

Theorem (4.1) [35]

Let $f(x) \in L_p \;\; X = [0,1]$, and L_n is positive linear operator, then

a) $\|L_n(f)\|_p \leq k\|f\|_{\sum n}$

b) $\|f - L_n(f)\|_p \leq C\tau_r\left(f; \sqrt[r]{(d_n)^s}\right)_p \qquad s \leq r$

where $d_n \leq min\left\{1, \left(\frac{a}{r}\right)^{\frac{r}{s}}\right\}$

4.3 Error estimate for (FFT) by L_n polynomial

Let $f(x)$ be a finite Fourier transform , the main theorem proved with some results.

Theorem (4.2)

Let $f(x)$ be an integrabal 1-periodic function on $[0,1]$ then $L_n(f)$ is a polynomials satisfies

$$\|f - L_n\|_p \le 6\,\tau_k\left(f, \sqrt{x(1-x)/n}\right)$$

Proof

$$f - L_n(f) = f - g(x) + g(x) - L_n(g(x)) + L_n(g(x)) - L_n(f)$$

Take the p-norm to both sides

$$\|f(x) - L_n(f)\|_p$$
$$= \left(\int_0^1 |[f - g(x) + g(x) - L_n(g(x)) + L_n(g(x)) - L_n(f)]|^p dx\right)^{\frac{1}{p}}$$

Using Minkowski inquality, get

$$\|f(x) - L_n(f)\|_p$$
$$\le \left(\int_0^1 |[f - g(x)]|^p dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_0^1 |[g(x) - L_n(g(x))]|^p dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_0^1 |[L_n(g(x)) - L_n(f)]|^p dx\right)^{\frac{1}{p}}$$

Since L_n is linear operator, then

$$\|f(x) - L_n(f)\|_p \leq \left(\int_0^1 |[f - g(x)]|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |[g(x) - L_n(g(x))]|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |[L_n(g(x) - f)]|^p dx\right)^{\frac{1}{p}}$$

Since $\|f - g(x)\|_p \leq C\tau_k(f,\delta)_p$ then

$$\left(\int_0^1 |[f - g(x)]|^p dx\right)^{\frac{1}{p}} = \|f - g(x)\|_p \leq C\tau_k(f,\delta)_p$$

By using theorem (1.4), (b) get

$$\left(\int_0^1 |[g(x) - L_n(g(x))]|^p dx\right)^{\frac{1}{p}} \leq \|L_n(g(x)) - g(x)\|_p \leq C\tau_r(f;\delta)_p$$

From (a)

$$\left(\int_0^1 |[L_n(g(x) - f)]|^p dx\right)^{\frac{1}{p}} = \|L_n(g(x) - f)\|_p \leq c\tau_k\left(f, \sqrt{x(1-x)/n}\right)$$

From theorem (2.7) get

$$\|f - L_n\|_p \leq 6\,\tau_k\left(f, \sqrt{x(1-x)/n}\right)$$

The following corollary to the theorem (4.2)

Corollary (4.3)

If f is a 1-periodic function on[0,1] with l 1-perodic derivatives then

$$\|f - L_n\|_p \le 6^{l+1} \frac{\tau^l\left(f, \frac{\sqrt{x(1-x)}}{n}\right)}{\left(\frac{\sqrt{x(1-x)}}{n}\right)^l}$$

Now, state and prove our results for 1-periodic function.

Theorem (4.4)

For $M \in N$ and f is 1-periodic function defined on [0,1] , let L_n^* by a polynomials ,then we have estimates

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \le 2\|L_n^* - f\|_p \qquad for\ |k| \le M$$

Proof

A simple calculation shows that for $|k| \le M$ we have

$$\left|\tilde{f}_{2M,k} - L_n^* + L_n^* - \hat{f}_{(k)}\right|$$

$$\le \left|\tilde{f}_{2M,k} - L_n^*\right| + \left|L_n^* - \hat{f}_{(k)}\right|$$

Then

$$\left|\tilde{f}_{2M,k} - L_n^*\right| = \left\|\int_0^1 \left(\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right) - L_n^*\left(\frac{j}{2M+1}\right)\right] e^{\frac{-2\pi ijk}{2M+1}}\right)^p dx\right\|^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left(\left|\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right) - L_n^*\left(\frac{j}{2M+1}\right)\right] e^{\frac{-2\pi ijk}{2M+1}}\right|\right)^p dx\right]^{\frac{1}{p}}$$

$$|\tilde{f}_{2M,k} - L_n^*| \leq \|f - L_n^*\|_p$$

And

$$|L_n^* - \hat{f}_{(k)}| = \left[\left|\int_0^1 \left(\int_0^1 [f(x) - L_n^*(x)]\, e^{-2\pi ikx}\right)^p dx\right|\right]^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left|\left(\int_0^1 [f(x) - L_n^*(x)]e^{-2\pi ikx}\right)^p\right| dx\right]^{\frac{1}{p}}$$

$$|\hat{f}(k) - L_n^*| \leq \|f - L_n^*\|_p$$

Then

$$|\tilde{f}_{2M,k} - L_n^* + L_n^* - \hat{f}_{(k)}| \leq |\tilde{f}_{2M,k} - L_n^*| + |\hat{f}_{(k)} - L_n^*|$$

$$|\tilde{f}_{2M,k} - \hat{f}\,(k)| \leq \|f - L_n^*\|_p + \|f - L_n^*\|_p$$

$$|\tilde{f}_{2M,k} - \hat{f}\,(k)| \leq 2\|L_n^* - f\|_p \qquad \blacksquare$$

Corollary (4.5)

Suppose that f is, 1-periodic function with $l \geq 0$ 1-periodic derivatives ; then

$$|\tilde{f}_{2M,k} - \hat{f}(k)| \leq 12\frac{6^l \tau_f^l\left(\frac{1}{2M\pi}\right)}{(2\pi M)^l}$$

For $|k| \leq M$

Proof

From theorem (4.4)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|L_n^* - f\|_p$$

Then from corollary (4.3)

$$\|f - L_n^*\|_P \leq 6^{l+1} \frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Then

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2.6^{l+1} \frac{\tau_f^l\left(\frac{1}{2M\pi}\right)}{(2M\pi)^l}$$

$$\leq 12 \frac{6^l \tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Corollary (4.6)

Suppose that f is 1-periodic function then

$$\left|\tilde{f}_{2M,k}\right| \leq \left|\hat{f}(k)\right| + 6\tau_f\left(\frac{1}{2\pi M}\right)$$

Proof

By theorem (4.4)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|f - L_n^*\|_P$$

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq \left|\tilde{f}_{2M,k} - L_n^* + L_n^* - \hat{f}(k)\right|$$

$$\leq \left|\tilde{f}_{2M,k} - L_n^*\right| + \left|L_n^* - \hat{f}(k)\right|$$

$$\left|\tilde{f}_{2M,k} - L_n^*\right| \leq \|f - L_n^*\|_p$$

Then from theorem (4.2)

$$\|f - L_n^*\|_p \leq 6\,\tau_f\left(\frac{1}{2\pi M}\right)$$

Then

$$\left|\tilde{f}_{2M,k}\right| \leq \|f - L_n^*\|_p + \left|\hat{f}(k)\right|$$

$$\left|\tilde{f}_{2M,k}\right| \leq 6\,\tau_f\left(\frac{1}{2\pi M}\right) + \left|\hat{f}(k)\right|$$ ∎

Theorem (4.7)

There is a universal constant C such that ,if f is 1-periodic function ,then

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq C \log M\, \|f - L_n^*\|_p$$

Proof

Let Dirichlet kernel

$$f_{2M}(x) = \int_0^1 D_M\,(x - y) f(y)dy$$

Then

$$\tilde{f}_{2M}(x) = \frac{1}{2M+1}\sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right) f\left(\frac{j}{2M+1}\right)$$

We observe that;

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq \left|\tilde{f}_{2M(x)} - L_n^*(x)\right| + \left|L_n^*(x) - f_{2M(x)}\right|$$

$$\leq \left\| \int_0^1 \left(\frac{1}{2M+1} \sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right) - L_n^*\left(\frac{j}{2M+1}\right)\right] + \int_0^1 D_M(x-y)[L_n^*(y) - f(y)]dy \right)^p dx \right\|^{\frac{1}{p}}$$

$$\leq \left[\int_0^1 \left| \left(\frac{1}{2M+1} \sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right) - L_n^*\left(\frac{j}{2M+1}\right)\right] + \int_0^1 D_M(x-y)[L_n^*(y) - f(y)]dy \right)^p \right| dx \right]^{\frac{1}{p}}$$

$$\leq \|f - L_n^*\|_p \cdot \left[\frac{1}{2M+1} \sum_{j=0}^{2M} \left| D_M\left(x - \frac{j}{2M+1}\right) \right| + \int_0^1 |D_M(x-y)|\, dy \right]$$

Both the sum and the integral in the last line are bounded by a constant times log M ∎

Corollary (4.8)

If f is 1-periodic function whose averaged modulus satisfies

$$\tau_f(\delta) = o(|\log \delta|^{-1}) \qquad (a)$$

Then, the (FFT) partial sum $(\tilde{f}_{2M})$ converges to f on [0,1] if f has l periodic derivatives and τ_f^l satisfies the above estimate, then for each $1 \leq j \leq l$,the sequence$\langle \partial_x^i \tilde{f}_{2M} \rangle$ converges to ∂_x^jf

Proof

Let p*∈ T_M be a best approximation to f ,we will use the triangle inquality to conclude that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq \left|\tilde{f}_{2M(x)} - L_n^*(x)\right| + \left|L_n^*(x) - f_{2M(x)}\right|$$

Applying theorem (4.2),we see that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq (C\log M + 6)\tau_f\left(\frac{1}{2M\pi}\right) \qquad (b)$$

The estimate in (a) implies the right-hand side of equation (b) tends to zero as M tends to infinity. The last inquality tell us that

$\left\|\tilde{f}_{2M(x)} - f_{2M(x)}\right\|$ tends to zero as M tends to infinity. ■

Conclusions and Recommendations for Future Work

The degree of the best approximation have been studied to estimate the error for finite Fourier transform ,bounded function ,integrable function and 1-perodic function in $L_p - spaces$ by positive linear operators in terms of averaged modulus of smoothness . the error estimate in $L_p - spaces$ have been calculate by :

1. The error estimate by trigonometric polynomial on [0,1] in $L_p - spaces$ in terms of averaged modulus of smoothness have been found .

2.The error estimate for finite Fourier transform via Valle-poussin operator on [0,1] in $L_p - spaces$ have been found .

3.The error estimate by L_n polynomial for finite Fourier transform in terms of averaged modulus of smoothness have been found.

Suggested many problems for future works.

1.The error estimate for the function may be any other periodic or a periodic function.

2.The error estimate for unbounded functions.

3. The error estimate for functions in $L_\infty - spaces$.

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المستخلص

هذه الرسالة تدرس التقريب لتحويل فورييه المنتهي للدالة (القابلة للتكامل ،المقيدة و الدورية) المعرفة على الفترة [0,1] في الفضاء L_p, $(1 \leq p < \infty)$ وتعاملنا مع تقدير الخطأ لتحويل فورييه المنتهي باستخدام مؤثرات خطية موجبة بواسطة معدل مقياس النعومة.

درست الرسالة أفضل تقريب لتحويل فورييه المنتهي في الفضاء L_p, $(1 \leq p < \infty)$ باستخدام متعددة حدود مثلثيه، مؤثر فاليه بوسون و متعددة حدود L_n على الفترة [0,1] بواسطة معدل مقياس النعومة.

النتائج التالية درست:

- قدر الخطأ لتحويل فورييه المنتهي للدالة (القابلة للتكامل، المقيدة و الدورية) على الفترة [0,1] في الفضاء L_p, $(1 \leq p < \infty)$ باستخدام متعددة حدود مثلثيه بواسطة معدل مقياس النعومه.

- قدر الخطأ لتحويل فورييه المنتهي للدالة (القابلة للتكامل، المقيدة و الدورية) على الفترة [0,1] في الفضاء L_p, $(1 \leq p < \infty)$ باستخدام متعددة الحدود L_n بواسطة معدل مقياس النعومه.

- قدر الخطأ لتحويل فورييه المنتهي للدالة (القابلة للتكامل، المقيدة و الدورية) على الفترة [0,1] في الفضاء L_p, $(1 \leq p < \infty)$ باستخدام مؤثر فاليه بوسون بواسطة معدل مقياس النعومه.

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وزارة التعليم العالي والبحث العلمي
الجامعة المستنصرية
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في الرياضيات

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أيلول - 2016