Integration :-

A function F is called an antiderivative of a function f if the derivative of F is f i.e:- $\frac{d}{dx}(F(x)) = f(x)$

and thus :- If $F(x)$ is any antiderivative of $f(x)$, then for any value of C, the function $F(x) + C$ is also an antiderivative of $f(x)$; moreover on any interval, every antiderivative of $f(x)$ is expressible in the form $F(x)$ plus a constant.

The process of finding antiderivatives is called antidifferentiation or "Integration", if.

$$\frac{d}{dx}[F(x)] = f(x)$$

then the function $F(x) + C$ are the antiderivatives of $f(x)$.

we denoted this by :-

$$\int f(x)\,dx = F(x) + C$$

which read as : "The indefinite integral of $f(x)$ equals $F(x)$ plus C"
where the symbol $\int$ is called an integral sign, C is called the constant of integral.

The symbol dx in the differentiation operation $\frac{d}{dx}[\ ]$ and in the antidifferentiation operation $\int[\ ]dx$ serves to identify the independent variable. For example $\int f(t)\,dt$ denotes a function of t.

Ex:-

Derivative formula	Equivalent Integration Formula
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

Integration Formulas:-

Defferentiation Formula	Integration Formula.
$\frac{d}{dx}[x] = 1$	$\int 1\, dx = x + c$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \ (r \neq -1)$	$\int x^r dx = \frac{x^{r+1}}{r+1} + c \ ; (r \neq -1)$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + c$
$\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\, dx = -\cos x + c$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + c$
$\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\, dx = -\cot x + c$
$\frac{d}{dx}[\sec x] = \sec x . \tan x$	$\int \sec x . \tan x\, dx = \sec x + c$
$\frac{d}{dx}[\csc x] = \csc x . \cot x$	$\int \csc x . \cot x\, dx = -\csc x + c$

Ex:-

$$\int x^2 dx = \frac{x^3}{3} + c$$

$$\int x^3 dx = \frac{x^4}{4} + c$$

$$\int \frac{1}{x^6} dx = \int x^{-6} dx = \frac{x^{-5}}{-5} + c = -\frac{1}{5x^5} + c$$

$$\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{x^{3/2}}{3/2} + c = \frac{2}{3} x^{3/2} + c = \frac{2}{3}\sqrt{x^3} + c$$

$$= \frac{2}{3}(\sqrt{x})^3 + c$$

Remarks:-

① $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$

② $\int c\,f(x)\,dx = c\int f(x)\,dx$; c is constant

③ $\int [f(x) \mp g(x)]\,dx = \int f(x)\,dx \mp \int g(x)\,dx$.

From ② and ③ we get:-

$$\int [c_1 f_1(x) \mp c_2 f_2(x) \mp \cdots \mp c_n f_n(x)]\,dx$$

$$= \int c_1 f_1(x)\,dx \mp \int c_2 f_2(x)\,dx \mp \cdots \mp \int c_n f_n(x)\,dx$$

$$= c_1\int f_1(x)\,dx \mp c_2\int f_2(x)\,dx \mp \cdots \mp c_n\int f_n(x)\,dx.$$

Ex:- Evaluate $\int 4\cos x\,dx$ and $\int (x+x^2)\,dx$.

Sol:

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$$

$$\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx = \left[\frac{x^2}{2}+c_1\right] + \left[\frac{x^3}{3}+c_2\right]$$

$$= \frac{1}{2}x^2 + \frac{1}{3}x^3 + C \qquad ; c_1 + c_2 = C$$

Ex:- Evaluate $\int (3x^6 - 2x^2 + 7x + 1)\,dx$

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int dx$$

$$= \frac{3}{7}x^7 - \frac{2}{3}x^3 + \frac{7}{2}x^2 + x + C$$

Ex:- Evaluat $\int \frac{\cos x}{\sin^2 x} dx$

Sol:-

$$\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} dx$$

$$= \int \csc x . \cot x \, dx = -\csc x + C$$

Ex:- Evaluat $\int \frac{t^2 - 2t^4}{t^4} dt$

$$\int \frac{t^2-2t^4}{t^4} dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2) dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C.$$

Exercise:-

In Exercises 1–30 evaluate the integrals and cheak your ~~re~~suls by differentiating the answers:-

① $\int x^8 dx$ ② $\int \frac{1}{x^5} dx$ ③ $\int x^{5/7} dx$

④ $\int \sqrt[3]{x^2} dx$ ⑤ $\int \frac{4}{\sqrt{t}} dt$ ⑥ $\int \frac{1}{2x^3} dx$

⑦ $\int x^3 \sqrt{x} \, dx$ ⑧ $\int (u^3 - 2u + 7) du$ ⑨ $\int (x^{-3} + \sqrt{x} - 3x^{1/4} + x^2) dx$

⑩ $\int \left(\frac{7}{y^{3/4}} - \sqrt[3]{y} + 4\sqrt{y}\right) dy$ ⑪ $\int (x^{2/3} - 4x^{-1/5} + 4) dx$

⑫ $\int (2 + y^2)^2 dy$ ⑬ $\int x(1 + x^3) dx$ ⑭ $\int \left(\frac{1-2t^3}{t^3}\right) dt$

⑮ $\int (1+x^2)(2-x) dx$ ⑯ $\int x^{1/3}(2-x)^2 dx$ ⑰ $\int \frac{x^5 + 2x^2 - 1}{x^4} dx$

⑱ $\int \left[\frac{1}{t^2} - \cos t\right] dt$ ⑲ $\int [4\sin x + 2\cos x] dx$ ⑳ $\int \frac{dy}{\csc y}$

3-5

(21) $\int [4\sec^2 x + \csc x \cot x)\, dx$ (22) $\int [\sqrt{\theta} - \csc^2\theta]\, d\theta$

(23) $\int \sec x [\sec x + \tan x]\, dx$ (24) $\int \sec x (\tan x + \cos x)\, dx$

(25) $\int \frac{\sin x}{\cos^2 x}\, dx$ (26) $\int \frac{\sin 2x}{\cos x}\, dx$ (27) $\int [\phi + \frac{2}{\sin^2\phi}]\, d\phi$

(28) $\int [1 + \sin^2\theta \csc\theta]\, d\theta$ (29) $\int \frac{\cos^3\theta - 5}{\cos^2\theta}\, d\theta$

(30) $\int \frac{x^2 \sin x + 2\sin x}{2 + x^2}\, dx$

(31) Find $f(x)$ if $\int f(x)\, dx = 5x^3 - 3x + C$

(32) Find $g(x)$ if $\int g(x)\, dx = \frac{1}{\sqrt{4-t^2}} + C$

(33) Evaluat $\int \tan^2 x\, dx$ and $\int \cot^2 x\, dx$

(33) sol $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

$\int \cot^2 x\, dx = \int (\csc^2 x - 1)\, dx = -\cot x - x + C$

Integral by Substitution :-

The method of substitution hinges on the following formula in which u stand for a differentiable function of x.

$$\int \left[f(u) \frac{du}{dx} \right] dx = \int f(u)\, du$$

Ex:- Evaluate $\int (x^2+1)^{50} \cdot 2x\, dx$

Sol:-

let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x$

then:-

$$\int (x^2+1)^{50}\, 2x\, dx = \int \left[u^{50} \frac{du}{dx} \right] dx = \int u^{50}\, du$$

$$= \frac{u^{51}}{51} + c = \frac{(x^2+1)^{51}}{51} + c$$

Integral by substitution:

step ① make choice for u say $u = g(x)$

step ② compute $du/dx = g'(x)$

step ③ make the substitution $u = g(x)$, $du = g'(x)\,dx$

At this stage, the entir integral must be interms of u; no x's should remain. If this is not the case, try a different choice of u.

step ④ Evaluate the resulting integral

step ⑤ Replace u by $g(x)$, so the final answer is interm of x.

Examples- Evaluate $\int \sin^2 x \cos x\, dx$

Sol:-

Let $u = \sin x \Rightarrow du = \cos x\, dx$

$$\therefore \int \sin^2 x \cos x\, dx = \int u^2\, du = \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c.$$

3-7

Ex:- Evaluat $\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$.

Sol:-

Let $u=\sqrt{x} \Rightarrow \frac{du}{dx}=\frac{1}{2\sqrt{x}} \Rightarrow du=\frac{1}{2\sqrt{x}}dx$

$\therefore \int \frac{\cos\sqrt{x}}{2\sqrt{x}}dx=\int \cos u\,du=\sin u+c=\sin\sqrt{x}+C$.

Ex:- Evaluate:- $\int 3x^2\sqrt{x^3+1}\,dx$

Let $u=x^3+1 \Rightarrow \frac{du}{dx}=3x^2 \Rightarrow du=3x^2dx$

$\therefore \int 3x^2\sqrt{x^3+1}\,dx=\int\sqrt{x^3+1}\cdot 3x^2dx=\int u^{1/2}du$

$=\frac{u^{3/2}}{3/2}+c=\frac{2}{3}(x^3+1)^{3/2}+C$.

Ex:- Evaluate:- $\int \sin(x+9)\,dx$

Let $u=x+9 \Rightarrow \frac{du}{dx}=1 \Rightarrow du=dx$

$\therefore \int \sin(x+9)\,dx=\int \sin u\,du=\cos u+C=\cos(x+9)+C$.

Ex:- Evaluate:- $\int \cos 5x\,dx$

$\int \cos 5x\,dx=\int \cos u\cdot\frac{1}{5}du=\frac{1}{5}\int\cos u\,du$; $u=5x \Rightarrow du$

$\Rightarrow du=5dx$

$=\frac{1}{5}\sin u+C=\frac{1}{5}\sin 5x+C$. $\Rightarrow dx=\frac{1}{5}du$

Ex:- Evaluate:- $\int \frac{dx}{(\frac{1}{3}x-8)^5}=$

Let $u=\frac{1}{3}x-8 \Rightarrow \frac{du}{dx}=\frac{1}{3} \Rightarrow dx=3du$

$\therefore \int \frac{dx}{(\frac{1}{3}x-8)^5}=\int\frac{3du}{u^5}=\int 3u^{-5}du=\frac{-3}{4}u^{-4}+c=-\frac{3}{4}(\frac{1}{3}x-8)^{-4}+C$

Ex:- Evaluate $\int (x+\sec^2 \pi x)\,dx$

$$\int (x+\sec^2 \pi x)\,dx = \int x\,dx + \int \sec^2 \pi x\,dx$$

$$= \frac{1}{2}x^2 + \int \sec^2 \pi x\,dx \qquad u = \pi x$$

$$du = \pi\,dx$$

$$= \frac{1}{2}x^2 + \int \sec^2 u \cdot \frac{1}{\pi}\,du \qquad dx = \frac{1}{\pi}\,du$$

$$= \frac{1}{2}x^2 + \frac{1}{\pi}\int \sec^2 u\,du = \frac{1}{2}x^2 + \frac{1}{\pi}\tan u + C$$

$$= \frac{1}{2}x^2 + \frac{1}{\pi}\tan \pi x + c.$$

Ex:- Evaluate:- $\int t^4 \sqrt[3]{3-5t^5}\,dt$

$$\therefore \int t^4 \sqrt[3]{3-5t^5}\,dt = \int \sqrt[3]{3-5t^5}\cdot t^4 dt \qquad u = 3-5t^5 \Rightarrow \frac{du}{dx} = -25t^4$$

$$= -\frac{1}{25}\int \sqrt[3]{u}\,du \qquad \Rightarrow dx = -\frac{1}{25}\,du$$

$$= -\frac{1}{25}\int u^{1/3}\,du = -\frac{1}{25}\,\frac{u^{4/3}}{4/3} + c = \frac{-3}{100}\,u^{4/3} + c$$

$$= \frac{-3}{100}\,(3-5t^5)^{4/3} + C.$$

Ex:- Evaluate $\int x^2\sqrt{x-1}\,dx$

let $u = x-1 \Rightarrow du = dx$.

$$\Rightarrow x = u+1 \Rightarrow x^2 = (u-1)^2 = u$$

$$\therefore \int x^2\sqrt{x-1}\,dx = \int (u+1)^2\sqrt{u}\,du = \int (u^2+2u+1)\sqrt{u}\,du$$

$$= \int (u^{5/2} + u^{3/2} + u^{1/2})\,du = \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.$$

3-9

Exercise :-

In exercise 1–16 evaluat the integrals by making the indicated substitutions:-

1. $\int 2x(x^2+1)^{23}\,dx \;;\; u=x^2+1$
2. $\int \cos^3 x \sin x\,dx \;;\; u=\cos x$
3. $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx \;;\; u=\sqrt{x}$
4. $\int \frac{3x\,dx}{\sqrt{4x^2+5}} \;;\; u=4x^2+5$
5. $\int \sec^2(4x+1)\,dx \;;\; u=4x+1$
6. $\int y\sqrt{1+2y^2}\,dy \;;\; u=1+2y^2$
7. $\int \sqrt{\sin\pi\theta}\,\cos\pi\theta\,d\theta \;;\; u=\sin\pi\theta$
8. $\int (2x+7)(x^2+7x+3)^{4/5} \;;\; u=x^2+7x+3$
9. $\int \cot x \csc^2 x\,dx \;;\; u=\cot x$
10. $\int (1+\sin t)^9 \cos t\,dt \;;\; u=1+\sin t$
11. $\int \cos 2x\,dx \;;\; u=2x$
12. $\int x\sec^2 x^2\,dx \;;\; u=x^2$
13. $\int x^2\sqrt{1+x}\,dx \;;\; u=1+x$
14. $\int [\csc(\sin x)]^2 \cos x\,dx \;;\; u=\sin x$
15. $\int \sin(x-\pi)\,dx \;;\; u=\pi-x$
16. $\int \frac{5x^4}{(x^5+1)^2}\,dx \;;\; u=x^5+1$

Solution of 7

$u=\sin\pi\theta \Rightarrow \frac{du}{d\theta}=\cos\pi\theta\cdot\pi \Rightarrow \cos\pi\theta\,d\theta=\frac{1}{\pi}du$

$\therefore \int\sqrt{\sin\pi\theta}\,\cos\pi\theta\,d\theta=\int\sqrt{u}\cdot\frac{1}{\pi}du=\frac{1}{\pi}\int u^{1/2}\,du$

$=\frac{1}{\pi}\frac{u^{3/2}}{3/2}+c=\frac{2}{3\pi}u^{3/2}+c=\frac{2}{3\pi}\left(\sqrt{\sin\pi\theta}\right)^3+c$

Sol of 9 $\quad u=\cot x \Rightarrow du=-\csc^2 x\,dx \Rightarrow -du=\csc^2 x\,dx$

$\therefore \int \cot x\csc^2 x\,dx=-\int u\,du=-\frac{u^2}{2}+c=-\frac{1}{2}\cot^2 x+c.$

In exercises 17 - 42 Evaluate the integrals by making appropriate substitution.

17. $\int x(2-x^2)^3\,dx$
18. $\int (3x-1)^5\,dx$
19. $\int \cos 8x\,dx$
20. $\int \sin 3x\,dx$
21. $\int \sec 4x \tan 4x\,dx$
22. $\int \sec^2 5x\,dx$
23. $\int t\sqrt{7t^2+12}\,dt$
24. $\int \frac{x}{\sqrt{4-5x^2}}\,dx$
25. $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$
26. $\int \frac{1}{(1-3x)^2}\,dx$
27. $\int \frac{x}{(4x^2+1)^3}\,dx$
28. $\int x\cos(3x^2)\,dx$
29. $\int \frac{\sin(5/x)}{x^2}\,dx$
30. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$
31. $\int x^2\sec^2(x^3)\,dx$
32. $\int \cos^3 2t \sin 2t\,dt$
33. $\int \sin^5 3t \cos 3t\,dt$
34. $\int \frac{\sin 2\theta}{(5+\cos 2\theta)^3}\,d\theta$
35. $\int \cos 4\theta\sqrt{2-\sin 4\theta}\,d\theta$
36. $\int \tan^3 5x \sec^2 5x\,dx$
37. $\int \sec^3 2x \tan 2x\,dx$
38. $\int [\sin(\sin\theta)]\cos\theta\,d\theta$
39. $\int x\sqrt{x-3}\,dx$
40. $\int \frac{y\,dy}{\sqrt{y+1}}$
41. $\int \sin^3 2\theta\,d\theta$
42. $\int \sec^4 3\theta\,d\theta$.

Sol of 27 Let $u=4x^2+1 \Rightarrow \frac{du}{dx}=8x \Rightarrow \frac{1}{8}du = x\,dx$

$$\therefore \int \frac{x\,dx}{(4x^2+1)^3} = \int \frac{1}{u^3}\cdot\frac{1}{8}du = \frac{1}{8}\int u^{-3}\,du = \frac{1}{8}\cdot\frac{u^{-4}}{-4}+c$$

$$= -\frac{1}{2}(4x^2+1)^{-4}+c = \frac{-1}{2(4x^2+1)^4}+c$$

Solution of 38

let $u=\sin\theta \Rightarrow du = \cos\theta\,d\theta$

$$\therefore \int [\sin(\sin\theta)]\cos\theta\,d\theta = \int \sin u\cdot du = -\cos u + c$$

$$= -\cos(\sin\theta)+c.$$

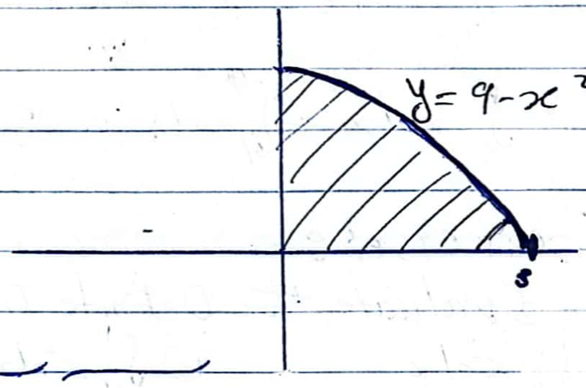
3-11

The Definite Integral :-

IF f is nonnegative and Continuous on $[a,b]$, then :-

$$A = \begin{bmatrix} \text{area under} \\ y = f(x) \\ \text{over } [a,b] \end{bmatrix} = \int_a^b f(x)\,dx$$

Ex:- Find the area under the Curve $y = 9 - x^2$ over the interval $[0, 3]$.



Sol:-

$$A = \int_0^3 (9 - x^2)\,dx$$

$$= 9x - \frac{x^3}{3} \Big|_0^3 = \underline{\underline{18}}$$

Theorem:- If f is continous on $[a,b]$ and if F is an antidrivative of f on $[a,b]$, then :-

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex:- evaluate :- $\int_1^2 x\,dx$

$$\int_1^2 x\,dx = \frac{1}{2} x^2 \Big|_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}.$$

Properties of Definite Integrals :

① $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

② $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

③ $\int_a^b [f_1(x) \mp f_2(x) \mp \cdots \mp f_n(x)]\,dx = \int_a^b f_1(x)\,dx \mp \int_a^b f_2(x)\,dx \mp \cdots \mp \int_a^b f_n(x)$

③ $\int_a^a f(x)\,dx = 0$

④ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Ex:- $\int_1^1 x^2\,dx = 0$

Ex:- $\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\Big|_0^4 = -8-0 = \underline{\underline{-8}}$

$\int_4^0 x\,dx = \int_4^0 x\,dx = \frac{x^2}{2}\Big|_4^0 = 0-\frac{16}{2} = \underline{\underline{-8}}$

Exercises:-

Evaluate the Definite Integrals:-

① $\int_2^3 x^3\,dx$ ② $\int_{-1}^1 x^2\,dx$ ③ $\int_{-1}^2 x(1+x^3)\,dx$

④ $\int_{-3}^0 (x^2-4x+7)\,dx$ ⑤ $\int_1^2 (t^2-2t+8)\,dt$ ⑥ $\int_0^1 (x^5-x^3+2x)\,dx$

⑦ $\int_1^3 \frac{1}{x^3}\,dx$ ⑧ $\int_1^2 \frac{1}{x^6}\,dx$ ⑨ $\int_1^2 \left(\frac{1}{x^3}-\frac{2}{x^2}+x^{-4}\right)dx$

⑩ $\int_{-2}^{-1} \left(u^{-4}+3u^{-2}-\frac{1}{u^5}\right)du$ ⑪ $\int_1^9 \sqrt{x}\,dx$ ⑫ $\int_1^4 x^{-3/5}\,dx$

⑬ $\int_4^9 2y\sqrt{y}\,dy$ ⑭ $\int_1^8 (5x^{2/3}-4x^{-2})\,dx$ ⑮ $\int_1^4 \left(\frac{3}{\sqrt{x}}-5\sqrt{x}-x^{-3/2}\right)dx$

⑯ $\int_4^9 (4y^{-1/2}+2y^{1/2}+y^{-5/2})\,dy$ ⑰ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta$ ⑱ $\int_0^{\frac{\pi}{4}} \sec^2\theta\,d\theta$

⑲ $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta\,d\theta$ ⑳ $\int_0^1 (x-\sec x\tan x)\,dx$

㉑ $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(x+\frac{2}{\sin^2 x}\right)dx$ ㉒ $\int_a^{4a} (a^{1/2}-x^{1/2})\,dx$; a is constant.

3-13

Evaluating Definite Integrals By Substitution :-

Ex:- $\int_0^2 2x(x^2+1)^3\,dx$

Sol:-

Let $u = x^2+1 \Rightarrow du = 2x\,dx$.

$$\therefore \int 2x(x^2+1)\,dx = \int u\,du = \frac{u^4}{4}+c = \frac{(x^2+1)^4}{4}+c$$

thus :-

$$\int_0^2 2x(x^2+1)^3\,dx = \left[\int 2x(x^2+1)^3\,dx\right]_{x=0}^{2} = \frac{(x^2+1)^4}{4}\Big|_0^2$$

$$= \frac{625}{4} - \frac{1}{4} = 156$$

هذه الطريقة تتم بإجراء التكامل ثم في النهاية تعويض حدود التكامل.

اما الطريقة الاخرى فتكون :-

$$\int_0^2 2x(x^2+1)^3\,dx.$$

$u = x^2+1 \Rightarrow du = 2x\,dx$.

$$\left.\begin{array}{l} \text{IF } x=0 \Rightarrow u=1 \\ \text{IF } x=2 \Rightarrow u=2^2+1=5 \end{array}\right\} \Rightarrow$$

$$\int_0^2 2x(x^2+1)^3\,dx = \int_1^5 u^3\,du = \frac{u^4}{4}\Big|_1^5 = \frac{625}{4} - \frac{1}{4} = 156.$$

اي أنه في هذه الطريقة يتم تغيير حدود التكامل اعتماداً على قيمة التعويض (اي قيمة u) للحصول على حدود جديدة للتكامل. علماً ان الطريقتين تعطي نفس الجواب:

Ex:- Evaluate $\int_0^{\pi/4} \cos(\pi - x)\,dx$

Sol:-

let $u = \pi - x \Rightarrow du = -dx$

$$\left.\begin{array}{l} \text{if } x=0 \Rightarrow u=\pi \\ \text{if } x=\frac{\pi}{4} \Rightarrow u=\frac{3\pi}{4} \end{array}\right\} \Rightarrow \int_0^{\frac{\pi}{4}} \cos(\pi-x)\,dx = \int_{\pi}^{\frac{3\pi}{4}} -\cos u\,du$$

$$= -\int_{\pi}^{\frac{3\pi}{4}} \cos u\,du = -\sin u\Big|_{\pi}^{\frac{3\pi}{4}} = \sin\frac{3\pi}{4}, -\sin\pi$$

$$= -1/\sqrt{2}.$$

Ex:- Evaluate $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$.

Sol:-

let $u = \sin 2x \Rightarrow du = 2\cos 2x \, dx \Rightarrow \frac{1}{2} du = \cos 2x \, dx$

If $x = 0 \Rightarrow u = \sin 0 = 0$

if $x = \frac{\pi}{8} \Rightarrow u = \sin \frac{\pi}{4} = 1/\sqrt{2}$

$$\therefore \int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx = \int_0^{1/\sqrt{2}} u^5 \cdot \frac{1}{2} du = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5 du$$

$$= \frac{1}{2} \frac{u^6}{6} \Big|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96}.$$

Exercises:-

Evaluate the integrals by substitutions

① $\int_1^2 (4x-2)^3 dx$ ② $\int_0^1 (2x+1)^4 dx$ ③ $\int_1^2 (4-3x)^8 dx$

④ $\int_{-1}^0 (1-2x)^3 dx$ ⑤ $\int_{-5}^{} x\sqrt{4-x} \, dx$ ⑥ $\int_0^8 x\sqrt{1+x} \, dx$

⑦ $\int_0^{\frac{\pi}{6}} 2\cos 3x \, dx$ ⑧ $\int_0^{2\pi} 4\sin(x/2) dx$ ⑨ $\int_{1-\pi}^{1+\pi} \sec^2\left(\frac{1}{4}x - \frac{1}{4}\right) dx$

⑩ $\int_{-2}^{-1} \frac{x}{(x^2+2)^3} dx$ ⑪ $\int_0^1 \sqrt[3]{a+bx} \, dx$; $b \neq 0$

⑫ $\int_0^1 \frac{du}{\sqrt{3u+1}}$ ⑬ $\int_1^2 \sqrt{5x+1} \, dx$ ⑭ $\int_{-1}^1 \frac{x^2 dx}{\sqrt{x^3+9}}$

⑮ $\int_{-1}^0 6t^2(t^3+1)^{19} dt$ ⑯ $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}} dx$ ⑰ $\int_1^2 \frac{dx}{x^2-6x+9}$

⑱ $\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} \sin x \cos x \, dx$ ⑲ $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sec^2 x \, dx$ ⑳ $\int_0^{\sqrt{\pi}} 5x\cos(x^2) dx$

㉑ $\int_0^{\frac{2\pi}{t}} t^2 \sin tx \, dx$; $t > 0$ ㉒ $\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x} \, dx$

㉓ $\int_{-\frac{\pi}{4}}^{\pi} \sin\theta \cos\theta \, d\theta$ ㉔ $\int_0^{\frac{\pi}{2}} \sin^2 3x \cos 3x \, dx$

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(25) $\int_0^{\frac{\pi}{4}} \frac{\cos 2x \, dx}{\sqrt{7-3\sin 2x}}$ (26) $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sec^2 3\theta \, d\theta$

(27) $\int_{-1}^{4} \frac{x\,dx}{\sqrt{5+x}}$ (28) $\int_0^1 \frac{y^2\,dy}{\sqrt{4-3y}}$

Solution of (6) :-

$$\int_0^8 x\sqrt{1+x}\,dx$$

let $u=1+x \Rightarrow du=dx$; if $x=0 \Rightarrow u=1$

$x=u-1$ if $x=8 \Rightarrow u=9$

$$\therefore \int_0^8 x\sqrt{1+x}\,dx = \int_1^9 (u-1)\cdot\sqrt{u}\,du = \int_1^9 (u^{\frac{3}{2}}-u^{\frac{1}{2}})\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\Big|_1^9 = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big|_1^9$$

$$= \frac{2}{5}(\sqrt{9})^5 - \frac{2}{3}(\sqrt{9})^3 - \frac{2}{5}(\sqrt{1})^5 + \frac{2}{3}(\sqrt{1})^3$$

$$= \frac{2}{5}*243 - \frac{2}{3}*27 - \frac{2}{5} + \frac{2}{3} = \frac{486}{5} - 18 - \frac{2}{5} + \frac{2}{3}$$

$$= \frac{486}{5} - 18 - \frac{2}{5} + \frac{2}{3} = \frac{1458-270-6+10}{15}$$

$$= \frac{1192}{15}$$

Solution of (9)

$$\int_{1-\pi}^{1+\pi} \sec^2(\tfrac{1}{4}x-\tfrac{1}{4})\,dx$$

let $u=\frac{1}{4}x-\frac{1}{4} \Rightarrow du=\frac{1}{4}dx \Rightarrow dx=4du$

if $x=1-\pi \Rightarrow u=\frac{1}{4}(1-\pi)-\frac{1}{4} = \cancel{\frac{1}{4}}-\frac{1}{4}\pi-\cancel{\frac{1}{4}} = -\frac{\pi}{4}$

if $x=1+\pi \Rightarrow u=\frac{1}{4}(1+\pi)-\frac{1}{4} = \cancel{\frac{1}{4}}+\frac{\pi}{4}-\cancel{\frac{1}{4}} = \frac{\pi}{4}$

$$\therefore \int_{1-\pi}^{1+\pi} \sec^2(\tfrac{1}{4}x-\tfrac{1}{4})\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 u\cdot 4du = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 u\,du$$

$$= 4\tan u\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 4\tan(\tfrac{\pi}{4}) - 4\tan(-\tfrac{\pi}{4}) = 4-(-4) = \underline{\underline{8}}$$

Solution of (16) $\int_1^3 \frac{(x+2)\,dx}{\sqrt{x^2+4x+7}}$

let $u = x^2+4x+7 \Rightarrow du = (2x+4)dx = 2(x+2)dx$

$\therefore \frac{1}{2}du = (x+2)\,dx$.

if $x=1 \Rightarrow u = (1)^2 + 4(1) + 7 = 12$

if $x=3 \Rightarrow u = (3)^2 + 4(3) + 7 = 9+12+7 = 28$

$$\int_1^3 \frac{(x+2)\,dx}{\sqrt{x^2+4x+7}} = \int_{12}^{28} \frac{du}{2\sqrt{u}} = \frac{1}{2}\int_{12}^{28} u^{-1/2}\,du = u^{1/2}\Big|_{12}^{28}$$

$$= \sqrt{28} - \sqrt{12} = 2\sqrt{7} - 2\sqrt{3}.$$

Solution of (20) $\int_0^{\sqrt{\pi}} 5x\cos(x^2)\,dx$

let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$.

$\therefore \begin{cases} \text{If } x=0 \Rightarrow u=0 \\ \text{if } x=\sqrt{\pi} \Rightarrow u=\pi \end{cases}$

$$\therefore \int_0^{\sqrt{\pi}} 5x\cos(x^2)\,dx = \int_0^{\pi} 5\cos u \cdot \frac{1}{2}du = \frac{5}{2}\int_0^{\pi}\cos u\,du$$

$$= \frac{5}{2}\sin u\Big|_0^{\pi} = \frac{5}{2}(\sin\pi - \sin 0) = 0$$

Remmark:- If $f(x)$ is Continuous then:-

① $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx \; ; \; a \leq c \leq b$.

② $\int_a^b f(x)dx \geq 0$ where $f(x) \geq 0$

③ $\int_a^b f(x)dx \geq \int_a^b g(x)dx \; ; \; f(x) \geq g(x)$

④ $\int_a^x f(t)dt = F(x)$ where $\int f(x) = F(x)$

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Inverse Functions:-

If the functions f and g satisfy the two conditions:

$$f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g$$
$$g(f(x)) = x \quad \text{" " " " " " } f$$

then we say that f is an inverse of g and g is an inverse of f. We also say that f and g are inverse of one another.

Ex:- Let $f(x) = 2x$, $g(x) = \frac{1}{2}x$.

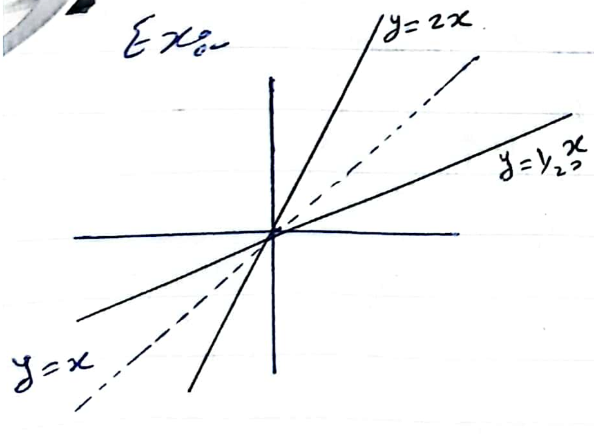
$$f(g(x)) = f(\tfrac{1}{2}x) = 2 \cdot (\tfrac{1}{2}x) = x \quad \forall x \in D_g$$
$$g(f(x)) = g(2x) = \tfrac{1}{2}(2x) = x \quad \forall x \in D_f$$

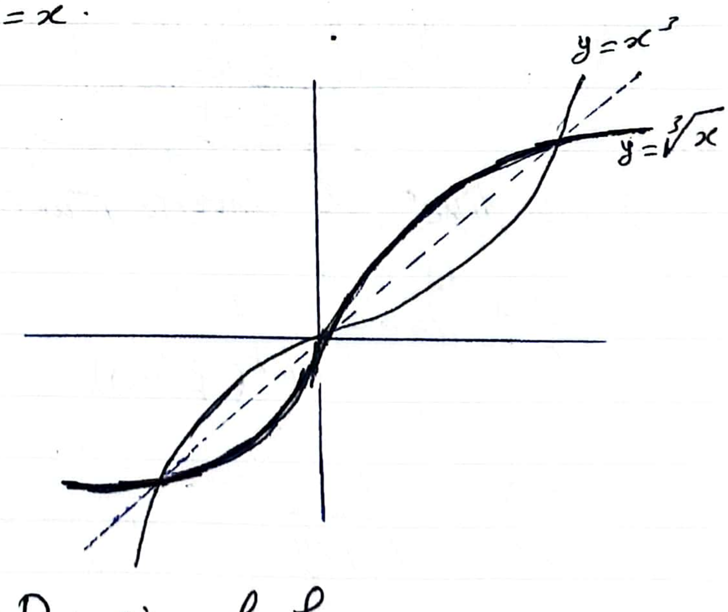
$\therefore$ f, g are inverses of one another.

remark:- ① The inverse of a function f is commonly denoted by f^{-1} and the symbol f^{-1} does not mean $1/f$.

② The graphs of any two inverse functions are reflections of one another about the line $y = x$.

Ex:-





③ Range of f^{-1} = Domain of f

Domain of f^{-1} = Range of f

Ex: Find the inverse of $f(x) = 4x-5$

$f(y) = 4y-5$

let $x = 4y-5 \Rightarrow y = \frac{1}{4}(x+5)$

$\therefore f^{-1}(x) = \frac{1}{4}(x+5)$.

$f(f^{-1}(x)) = 4\left(\frac{1}{4}(x+5)\right) - 5 = x+5-5 = x$.

$f^{-1}(f(x)) = \frac{1}{4}[(4x-5)+5] = x$

Ex:- Find the inverse of: $f(x) = \dfrac{x}{x-2}$

let $x = \dfrac{y}{y-2} \Rightarrow x(y-2) = y$

$\Rightarrow xy - 2x = y$

$\Rightarrow y - xy = -2x$

$\Rightarrow y(1-x) = -2x$

$\Rightarrow y = \dfrac{-2x}{1-x} = \dfrac{2x}{x-1}$

$\therefore f^{-1}(x) = \dfrac{2x}{x-1}$

Derivative of Inverse Functions:-

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

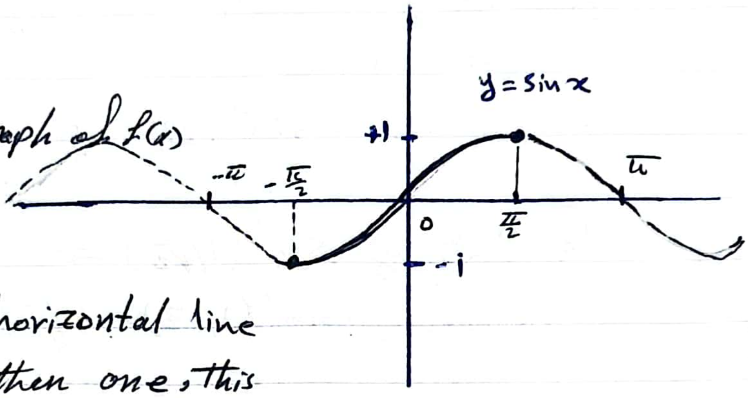
$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Inverse Trigonometric Function:-

Inverse Sine:-

let $f(x) = \sin x$

IF we have sketched the graph of $f(x)$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$:



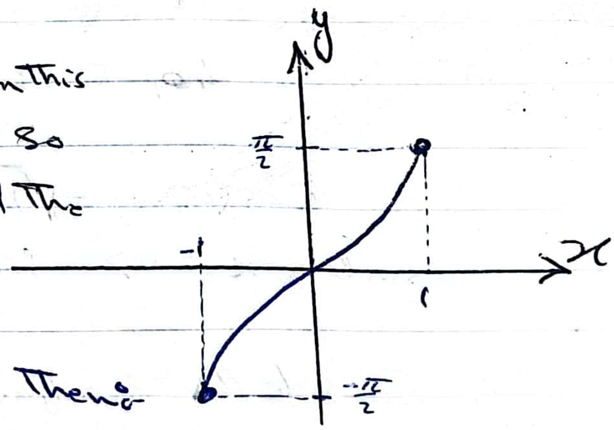
From the graph there is no horizontal line cuts the graph of $f(x)$ more then one, this function has inverse. This inverse is called the Inverse Sine Function and This denoted by $\sin^{-1}x$

notice the symbol $\sin^{-1}x$ is never used to denote $\frac{1}{\sin x}$.

Where $1/\sin x$ can be rewritten as $(\sin x)^{-1}$ or $\csc x$.

The graph of $\sin^{-1}x$ is obtained by reflecting the graph of $(\sin x)$ about The line $y = x$.

if $y = \sin^{-1}x$ Then it is evident, from This graph that $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ so The domain of $\sin^{-1}x$ is $[-1, 1]$ and The range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



IF $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ Then:

$y = \sin^{-1}x$ and $\sin y = x$

are equivalent.

i.e $\sin^{-1}(\sin y) = y$ if $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\sin(\sin^{-1}x) = x$ if $-1 \leq x \leq 1$

Ex:- Find ⓐ $\sin^{-1}(\frac{1}{2})$ ⓑ $\sin^{-1}(-1/\sqrt{2})$

Sol:-

ⓐ let $y = \sin^{-1}(\frac{1}{2}) \Rightarrow \sin y = \frac{1}{2}$; $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\Rightarrow y = \frac{\pi}{6}$

$\therefore \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

ⓑ let $y = \sin^{-1}(1/\sqrt{2}) \Rightarrow \sin y = \frac{1}{\sqrt{2}} \Rightarrow y = -\frac{\pi}{4}$

$\therefore \sin^{-1}(1/\sqrt{2}) = -\frac{\pi}{4}$

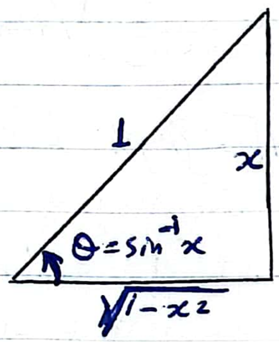
Ex:- Simplify The function $\cos(\sin^{-1}x)$.

Solution:-

Since $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \cos^2(\sin^{-1}x) = 1 - \sin^2(\sin^{-1}x)$

$\therefore |\cos(\sin^{-1}x)| = \sqrt{1 - \sin^2(\sin^{-1}x)}$



Since $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2} \Rightarrow \cos(\sin^{-1}x)$ is nonnegative.

then we can drop the absolute value sign and write.

$$\cos(\sin^{-1}x) = \sqrt{1-x^2}$$

$\sin\theta = x \quad \Rightarrow \quad \sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{1} = x$

$\therefore$ The side that is adjacent to θ has length $\sqrt{1-x^2}$

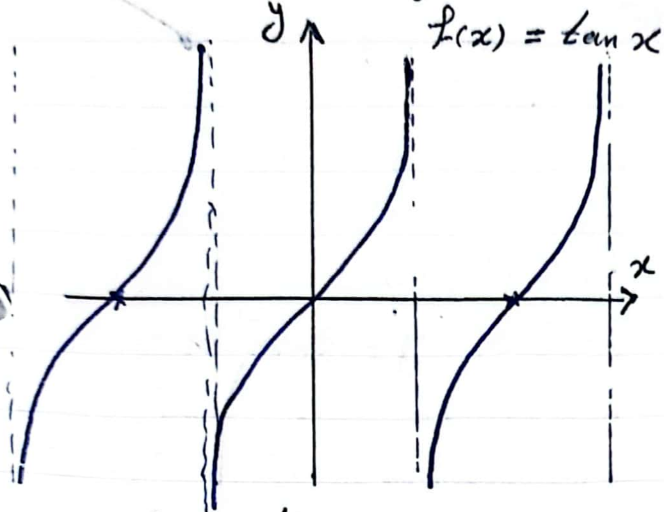
$\Rightarrow \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore \cos(\sin^{-1}x) = \sqrt{1-x^2}$

Inverse Tangent

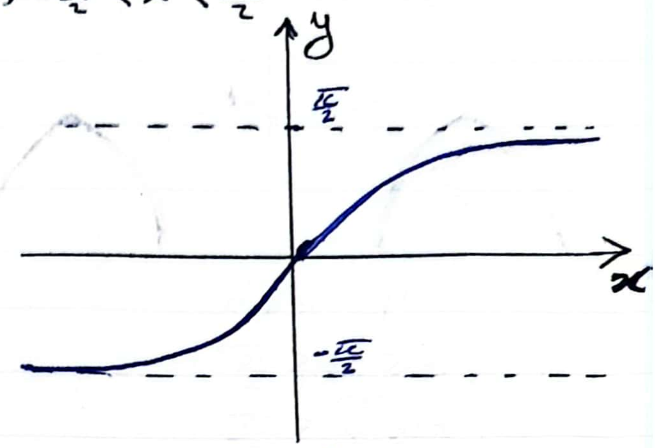
The Inverse Tangent function, $\tan^{-1}x$, is defined to be the inverse of the restricted tangent function :-

$f(x) = \tan x \quad ; -\frac{\pi}{2} < x < \frac{\pi}{2}$



$f(x) = \tan x$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1}x$

IF $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ then :-

$$y = \tan^{-1}x \quad \text{and} \quad \tan y = x$$

are equivalent.

i.e :-

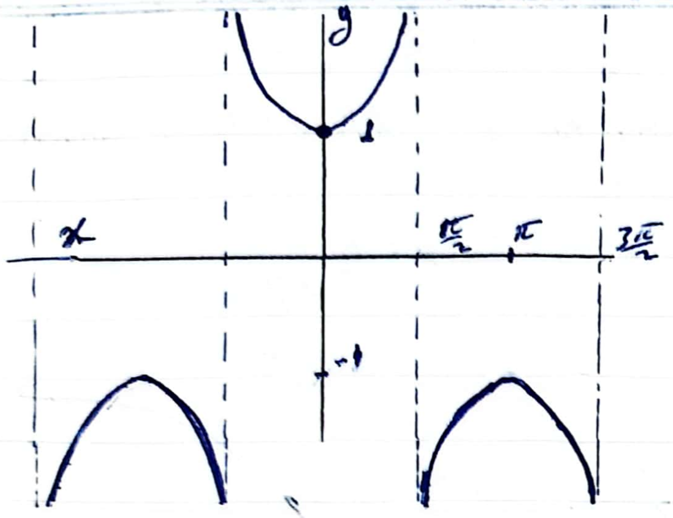
$$\tan^{-1}(\tan y) = y \quad \text{if } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\tan(\tan^{-1}x) = x \quad \text{if } -\infty < x < \infty$$

Inverse Secant :-

The Inverse Secant function, $\sec^{-1}x$ is defined to be the inverse of the restricted secant function :-

$$f(x) = \sec x \qquad 0 \leq x < \frac{\pi}{2} \text{ or } \pi \leq x < \frac{3\pi}{2}.$$

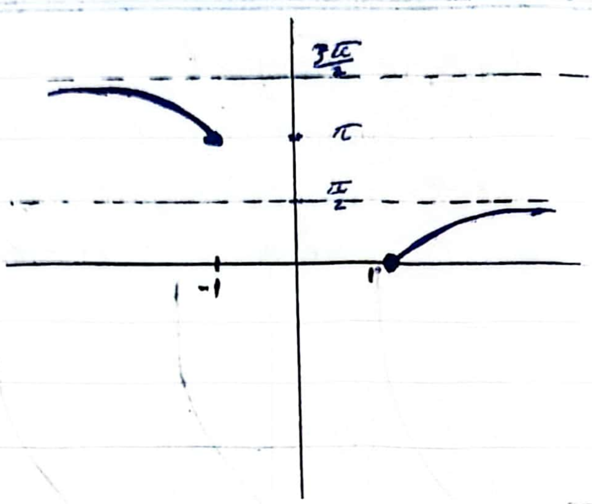


$y = \sec x$

$0 \leq x < \frac{\pi}{2}$

or

$\pi \leq x < \frac{3\pi}{2}$



$y = \sec^{-1} x$

IF $|x| \geq 1$ and if $0 \leq y < \frac{\pi}{2}$ or $\pi \leq y < \frac{3\pi}{2}$; Then

$y = \sec^{-1} x$ and $\sec y = x$

are equivalent.

i.e

$\sec^{-1}(\sec y) = y$ if $0 \leq y < \frac{\pi}{2}$ or $\pi \leq y < \frac{3\pi}{2}$

$\sec(\sec^{-1} x) = x$ if $x \leq -1$ or $x \geq 1$

Inverse Cosine, Cotangent and Cosecant:

$y = \cos^{-1} x \cong x = \cos y$ if $\begin{cases} 0 \leq y \leq \pi \\ -1 \leq x \leq 1 \end{cases}$

$y = \cot^{-1} x \cong x = \cot y$ if $\begin{cases} 0 < y < \pi \\ -\infty < x < \infty \end{cases}$

$y = \csc^{-1} x \cong x = \csc y$ if $\begin{cases} 0 < y \leq \frac{\pi}{2} \text{ or } -\pi < y \leq -\frac{\pi}{2} \\ |x| \geq 1 \end{cases}$

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

and

$$\sin^{-1}(-x) = -\sin^{-1}x$$
$$\tan^{-1}(-x) = -\tan^{-1}x$$
$$\sec^{-1}(-x) = \pi + \sec^{-1}x \quad \text{if } x \geq 1$$

Exercises :-

From 1 To 26 Find the exact Value of :-

1. $\sin^{-1}(-1)$
2. $\cos^{-1}(-1)$
3. $\tan^{-1}(-1)$
4. $\cot^{-1}(1)$
5. $\sec^{-1}(1)$
6. $\csc^{-1}(1)$
7. $\sin^{-1}(\frac{1}{2}\sqrt{3})$
8. $\cos^{-1}(1/2)$
9. $\tan^{-1}(1)$
10. $\cot^{-1}(-1)$
11. $\sec^{-1}(-2)$
12. $\csc^{-1}(-2)$
13. $\sin^{-1}(\sin\frac{\pi}{7})$
14. $\sin^{-1}(\sin\pi)$
15. $\sin^{-1}(\sin\frac{5\pi}{7})$
16. $\sin^{-1}(\sin 360)$
17. $\cos^{-1}(\cos\frac{\pi}{7})$
18. $\cos^{-1}(\cos\pi)$
19. $\cos^{-1}(\cos\frac{12\pi}{7})$
20. $\cos^{-1}(\cos 200)$
21. $\sec[\sin^{-1}(-\frac{3}{4})]$
22. $\sin[2\cos^{-1}(\frac{3}{5})]$
23. $\tan^{-1}[\sin(\frac{-\pi}{2})]$
24. $\sin^{-1}[\cot(\frac{\pi}{4})]$
25. $\sin[\sin^{-1}(\frac{2}{3}) + \cos^{-1}(\frac{1}{3})]$
26. $\tan[2\sec^{-1}(\frac{3}{2})]$

solution of

(27) Given that $\theta = \sin^{-1}(-\frac{1}{2}\sqrt{3})$, find the exact values of $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$.

(28) Given that $\theta = \cos^{-1}(\frac{1}{2})$, find the exact values of $\sin\theta$, $\tan\theta$, $\sec\theta$ and $\csc\theta$.

(29) Given that $\theta = \tan^{-1}(\frac{4}{3})$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$.

Solution

(30) Sketch the graphs of $\cos^{-1}x$, $\cot^{-1}x$ and $\csc^{-1}x$.

(31) Sketch the graphs of :-

(a) $y = \cos^{-1}\frac{1}{3}x$

(b) $y = 2\cot^{-1}2x$

(c) $y = \sin^{-1}2x$

(d) $y = \tan^{-1}\frac{1}{2}x$.

Derivatives and Integrations Involving Inverse Trigonometric Functions

$$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}[\sin^{-1}u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}[\cos^{-1}u] = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1+x^2} \Rightarrow \frac{d}{dx}[\tan^{-1}u] = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2} \Rightarrow \frac{d}{dx}[\cot^{-1}u] = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\sec^{-1}x] = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \frac{d}{dx}[\sec^{-1}u] = \frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{x\sqrt{x^2-1}} \Rightarrow \frac{d}{dx}[\csc^{-1}u] = \frac{-1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

Ex:- Find dy/dx if $y = \sin^{-1}(x^3)$

Sol:-

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}\cdot 3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

Ex:- Find dy/dx if $y = \sec^{-1}(x^2 - \frac{1}{2}x)$

Sol:-

$$\frac{dy}{dx} = \frac{1}{(x^2-\frac{1}{2}x)\sqrt{(x^2-\frac{1}{2}x)^2-1}}\cdot(2x-\tfrac{1}{2})$$

$$= \frac{\frac{1}{2}(4x-1)}{(x^2-\frac{1}{2}x)\sqrt{x^4-x^3+\frac{1}{4}x^2-1}} = \frac{4x-1}{2(x^2-\frac{1}{2}x)\sqrt{x^4-x^3+\frac{1}{4}x^2-1}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$$

Ex:- Evaluate $\int \frac{dx}{1+3x^2}$

Sol:-

let $u = \sqrt{3}x \Rightarrow du = \sqrt{3}\,dx \Rightarrow \frac{1}{\sqrt{3}} du = dx$

$$\therefore \int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}} \int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}} \tan^{-1} u + C = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + C$$

Ex: Evaluate $\int \frac{dx}{a^2+x^2}$; $a \neq 0$; a constant.

Let $x = au \Rightarrow dx = a\,du$.

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\,du}{a^2+a^2u^2} = \int \frac{a\,du}{a^2(1+u^2)} = \frac{1}{a} \int \frac{du}{1+u^2}$$

$$= \frac{1}{a} \tan^{-1} u + C = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

From above Ex. we get:-

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C \quad ;(a>0)$$

Exercises:-

In Exercises ① - ⑱ Find dy/dx.

① $y=\sin^{-1}(\frac{1}{3}x)$ ② $y=\cos^{-1}(2x+1)$ ③ $y=\tan^{-1}(x^2)$

④ $y=\cot^{-1}(\sqrt{x})$ ⑤ $y=\sec^{-1}(x^7)$ ⑥ $y=\csc^{-1}(4x)$

⑦ $y=(\tan x)^{-1}$ ⑧ $y=\dfrac{1}{\tan^{-1}x}$ ⑨ $y=\sin^{-1}(1/x)$

⑩ $y=\cos^{-1}(\cos x)$ ⑪ $y=\sqrt{\cot^{-1}x}$ ⑫ $y=x^2(\sin^{-1}x)^3$

⑬ $y=\sin^{-1}x+\cos^{-1}x$ ⑭ $y=\sec^{-1}x+\csc^{-1}x$ ⑮ $y=\tan^{-1}\left(\frac{1-x}{1+x}\right)$

⑯ $y=(1+x\csc^{-1}x)^{10}$ ⑰ $y=\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ ⑱ $y=\sin^{-1}(x^2\cos^{-1}x)$

Sol. of ⑧ $y=\dfrac{1}{(\tan^{-1}x)}$

$$y=(\tan^{-1}x)^{-1} = -(\tan^{-1}x)^{-2} * \frac{1}{1+x^2}$$

$$=\frac{-1}{(1+x^2)(\tan^{-1}x)^2}.$$

Sol. of ⑱ $y=\sin^{-1}(x^2\cos^{-1}x)$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(x^2\cos^{-1}x)^2}}\cdot\left[x^2\cdot\frac{-1}{\sqrt{1-x^2}}+2x\cos^{-1}x\right]$$

$$=\frac{\frac{-x^2}{\sqrt{1-x^2}}+2x\cos^{-1}x}{\sqrt{1-x^4(\cos^{-1}x)^2}}.$$

⑲ Find $\frac{dy}{dx}$ by implicit differentiation for:-

ⓐ $x^3+x\tan^{-1}y=\sec^{-1}y$ ⓑ $\sin^{-1}(xy)=\cos^{-1}(x-y)$

Sol. of ⓑ

$$\frac{1}{\sqrt{1-x^2y^2}}\cdot\left(x\frac{dy}{dx}-y\right)=\frac{-1}{\sqrt{1-(x-y)^2}}\cdot\left(1-\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{x\frac{dy}{dx}-y}{\sqrt{1-x^2y^2}}=\frac{\frac{dy}{dx}-1}{\sqrt{1-x^2+2xy-y^2}}$$

In Exercises (20) – (36) evaluat the integrals:-

(20) $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$ (21) $\int_{-1}^{1} \frac{dx}{1+x^2}$ (22) $\int \frac{dx}{1+16x^2}$

(23) $\int_{\sqrt{2}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$ (24) $\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$ (25) $\int \frac{dx}{\sqrt{1-4x^2}}$

(26) $\int \frac{dx}{x\sqrt{9x^2-1}}$ (27) $\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$ (28) $\int \frac{t}{t^4+1}\,dt$

(29) $\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$ (30) $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+1}$ (31) $\int \frac{dx}{x\sqrt{1-4x^2}}$

(32) $\int \frac{dx}{\sqrt{9-x^2}}$ (33) $\int \frac{dx}{5+x^2}$ (34) $\int \frac{dx}{x\sqrt{x^2-\pi}}$

(35) $\int \frac{dx}{\sqrt{9-4x}}$ (36) $\int \frac{dy}{y\sqrt{5y^2-3}}$.

Sol. of (24)