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Techniques of Integration

① Integration by Bart:-

$$\int \frac{d}{dx}[f(x)\cdot g(x)]\,dx = \int [f(x)\cdot g'(x) + g(x)\cdot f'(x)]\,dx$$

$$= \int f(x)\cdot g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$$

Then

$$f(x)\cdot g(x) + C = \int f(x)\,g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$$

$$\Rightarrow \int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\,dx + C$$

and thus :-

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx.$$

IF we set $u = f(x) \Rightarrow du = f'(x)\,dx$

$v = g(x) \Rightarrow dv = g'(x)\,dx$

i.e

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

IF the integration is definite we get:-

$$\boxed{\int_a^b u\,dv = u\cdot v\Big]_a^b - \int_a^b v\,du}$$

CamScanner ile tarandı

Ex: Evaluat $\int xe^x dx$

Sol:-

$u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = \int dv = \int e^x dx = e^x$

$\therefore \int \underset{u}{x} \underset{dv}{e^x dx} = u.v - \int v du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c$

notice :- If we use another choice i.e.

$u = e^x \Rightarrow du = e^x dx$

$dv = x \Rightarrow v = \frac{1}{2}x^2$

then

$\int xe^x dx = u.v - \int v du = \frac{1}{2}x^2 e^x - \frac{1}{2}\int x^2 e^x dx$?!

Ex:- Evaluat $\int x^2 e^{-x} dx$

let $u = x^2 \Rightarrow du = 2x dx$

$dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$

$\therefore \int x^2 e^{-x} dx = u.v - \int v du = -x^2 e^{-x} + 2\int x e^{-x} dx$

let $u_1 = x \Rightarrow du_1 = dx$

$dv_1 = +e^{-x} dx \Rightarrow v_1 = -e^{-x}$

$\therefore \int x e^{-x} dx = u_1.v_1 - \int v_1 du_1$

$= -xe^{-x} + \int e^{-x} dx$

$= -xe^{-x} - e^{-x} + c_1$

$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\int x e^{-x} dx$

$= -x^2 e^{-x} + 2[-xe^{-x} - e^{-x} + c_1] + c_2$

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C \quad ; C = 2c_1 + c_2$

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Ex: Evaluate $\int \ln x\, dx$

Solution:-

let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$

$dv = dx \Rightarrow v = \int dv = \int dx = x$.

$\therefore \int \ln x\, dx = u.v - \int v\, du = x \ln x - \int x \cdot \frac{1}{x}\, dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

Ex:- Evaluet $\int_0^1 \tan^{-1} x\, dx$.

Solution:-

let $u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2}\, dx$

$dv = dx \Rightarrow v = x$

$\therefore \int_0^1 \tan^{-1} x\, dx = u.v \Big|_0^1 - \int_0^1 v\, du$

$= x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$

$= x \tan^{-1} x \Big|_0^1 - \frac{1}{2} \ln |1+x^2| \Big]_0^1$

$= \tan^{-1} 1 - 0 - \frac{1}{2} \ln 2 + \frac{1}{2} \ln 1$

$= \frac{\pi}{4} - \ln \sqrt{2}$

Example:- Evaluate $\int e^x \cos x\, dx$.

Solution:- let $u = e^x \Rightarrow du = e^x dx$

$dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x$.

$\therefore \int e^x \cos x\, dx = u.v - \int v\, du = e^x \sin x - \int \underset{u_1}{e^x} \underset{v_1}{\sin x}\, dx$

$\therefore \int e^x \sin x\, dx = u_1 v_1 - \int v_1 du_1$ $\quad$ $\boxed{\begin{array}{l} u_1 = e^x \Rightarrow du = e^x dx \\ dv_1 = \sin x \Rightarrow v_1 = -\cos x \end{array}}$

$= -e^x \cos x + \int e^x \cos x$

$$\therefore \int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x$$

$$\Rightarrow$$

$$\Rightarrow \quad 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\therefore \int e^x \cos x\,dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$$

Ex:- Evaluate $\int x^2 e^x\,dx$.

Solution ①

if $u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^x dx \Rightarrow v = \int e^x\,dx = e^x$

$$\therefore \int x^2 e^x\,dx = u\cdot v - \int v\,du = x^2 e^x - \int 2x e^x\,dx$$

$$= x^2 e^x - 2\int \underbrace{x}_{u_1}\,\underbrace{e^x dx}_{dv_1} \qquad u_1 = x \Rightarrow du_1 = dx$$

$$= x^2 e^x - 2\left[u_1 v_1 - \int v_1 du_1\right] \qquad dv_1 = e^x dx \Rightarrow v_1 = e^x$$

$$= x^2 e^x - 2\left[x e^x - \int e^x dx\right]$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Solution ② Another Solution for this example and any Example need to applied u.dv more than one time by Tabular Integration which as show as:-

$$\int x^2 e^x\,dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

f(x) and its derivatives		g(x) and its integrate
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x

Ex:- Evaluate $\int x^3 \sin x\,dx$

diff.		integ.
x^3	+	$\sin x$
$3x^2$	−	$-\cos x$
$6x$	+	$-\sin x$
6	−	$+\cos x$
0		$+\sin x$

$$\int x^3 \sin x\,dx$$

$$= -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

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Exercises:-

Evaluate the Integrals:-

(1) $\int \ln(2x+3)\,dx$	(2) $\int x e^{3x}\,dx$	(3) $\int x \ln(x)\,dx$
(4) $\int x \ln\sqrt{x}\,dx$	(5) $\int \sin^{-1}x\,dx$	(6) $\int \cos^{-1}x\,dx$
(7) $\int x^2 e^{-2x}\,dx$	(8) $\int x^3 e^{-x}\,dx$	(9) $\int e^x \sin x\,dx$
A (10) $\int e^{-3\theta}\sin 3\theta\,d\theta$	B (11) $\int e^{2x}\cos 3x\,dx$	(12) $\int e^{ax}\sin bx\,dx$
(13) $\int \frac{\cos 2\pi x}{e^{2\pi x}}\,dx$	X (14) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$	(15) $\int \frac{x e^x}{(x+1)^2}\,dx$
(16) $\int x^2 \ln x\,dx$	(17) $\int x^2 \cos x\,dx$	(18) $\int x \tan^{-1}x\,dx$
(19) $\int x\sin(3x+1)\,dx$	(20) $\int x\sinh x\,dx$	(21) $\int x\sec^2 x\,dx$ X
(22) $\int \cos(\ln x)\,dx$	(23) $\int \sin(3\ln x)\,dx$	(24) $\int \sin(\ln x)\,dx$
(25) $\int (\ln x)^2\,dx$	(26) $\int x\tan^2 x\,dx$	(27) $\int_0^1 x e^{5x}\,dx$
(28) $\int_0^2 x e^{2x}\,dx$	(29) $\int_1^e x^2\ln x\,dx$	(30) $\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2}\,dx$
(31) $\int_{-2}^{2} \ln(x+3)\,dx$	(32) $\int_0^{1/2} \sin^{-1}x\,dx$	(33) $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
(34) $\int_1^2 x\sec^{-1}x\,dx$	(35) $\int_0^{\frac{\pi}{2}} x\sin 4x\,dx$	(36) $\int_0^{\pi}(x+x\cos x)\,dx$
(37) $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$	(38) $\int_0^2 \ln(x^2+1)\,dx$	(39) $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
(40) $\int_0^1 x e^{-\sqrt{x}}\,dx$	(41) $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx$	(42) $\int (r^2+r+1)e^r\,dr$
(43) $\int_0^1 x\sqrt{1-x}\,dx$	(44) $\int z(\ln z)^2\,dz$	(45) $\int_0^{\frac{\pi}{3}} x\tan^2 x\,dx$.

Solution of (12):- $\int e^{ax}\sin bx\,dx$

Sol:- let $u = e^{ax} \Rightarrow du = a e^{ax}dx$

$dv = \sin bx \Rightarrow v = \int \sin bx\,dx = \frac{-1}{b}\cos bx$

$$\therefore \int e^{ax}\sin bx\,dx = u\cdot v - \int v\,du = -\frac{1}{b}e^{ax}\cos bx + \int \frac{+1}{b}\cos bx\cdot a e^{ax}dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

let $u_1 = e^{ax} \Rightarrow du_1 = ae^{ax}dx$

$dv_1 = \cos bx\,dx \Rightarrow v = \int \cos bx\,dx = \frac{1}{b}\sin bx$

$$\therefore \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[u_1 \cdot v_1 - \int v_1 du_1\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[\frac{1}{b}e^{ax}\cdot\sin bx - \int \frac{1}{b}\sin bx \cdot ae^{ax}dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx + C$$

$$\int e^{ax}\sin bx\,dx + \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + C$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + C$$

$$\therefore \int e^{ax}\sin bx\,dx = \left(\frac{1}{1+\frac{a^2}{b^2}}\right)\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + C\right]$$

Solution of 15 $\int \frac{xe^x}{(x+1)^2}dx$ let $u = xe^x \Rightarrow du = (xe^x + e^x)dx$

$dv = \frac{1}{(x+1)^2}dx \Rightarrow v = \frac{-1}{(x+1)}$

$$\int \frac{xe^x}{(x+1)^2}dx = u.v - \int v\,du$$

$$= \frac{-xe^x}{x+1} - \int \frac{-1}{x+1}(xe^x + e^x)\,dx$$

$$= \frac{-xe^x}{x+1} + \int \frac{e^x\cancel{(x+1)}}{\cancel{(x+1)}}dx = \frac{xe^x}{x+1} + \int e^x\,dx$$

$$= \frac{-xe^x}{x+1} + e^x + C$$

4-15

Solution of (23) $\int \sin(3\ln x)\,dx$.

$u = \sin(3\ln x)\,dx \Rightarrow du = \cos(3\ln x)\cdot 3\cdot\frac{1}{x}\,dx$
$= \frac{3}{x}\cos(3\ln x)\,dx$

$dv = dx \Rightarrow v = x$.

$\therefore \int \sin(3\ln x)\,dx = u\cdot v - \int v\,du = x\sin(3\ln x) - \int x\cdot\frac{3}{x}\cos(3\ln x)\,dx$

$= x\sin(3\ln x) - 3\int \cos(3\ln x)\,dx$.

let $u_1 = \cos(3\ln x) \Rightarrow du_1 = \frac{-3}{x}\sin(3\ln x)\,dx$
$dv_1 = dx \Rightarrow v_1 = x$

$\therefore \int \sin(3\ln x)\,dx = x\sin(3\ln x) - 3\left[u_1v_1 - \int v_1\,du_1\right]$

$= x\sin(3\ln x) - 3\left[x\cos(3\ln x) - \int x\cdot\frac{-3}{x}\sin(3\ln x)\,dx\right]$

$= x\sin(3\ln x) - 3x\cos(3\ln x) - 9\int \sin(3\ln x)\,dx + c$

$10\int \sin(3\ln x)\,dx = x\sin(3\ln x) - 3x\cos(3\ln x) + c$

$\therefore \int \sin(3\ln x)\,dx = \frac{1}{10}x\sin(3\ln x) - \frac{3}{10}x\cos(3\ln x) + c$.

Solution of (29) $\int_1^e x^2\ln x\,dx$

Sol:-

let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$
$dv = x^2dx \Rightarrow v = \int x^2dx = \frac{1}{3}x^3$

$\therefore \int_1^e x^2\ln(x)\,dx = u\cdot v\,\Big|_1^e - \int_1^e v\,du = \frac{1}{3}x^3\ln x\,\Big|_1^e - \int_1^e \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$

$= (\frac{1}{3}e^3 - 0) - \frac{1}{3}\int_1^e x^2\,dx = \frac{1}{3}e^3 - \frac{1}{9}x^3\,\Big|_1^e$

$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9} = \frac{2}{9}e^3 + \frac{1}{9}$

4-16

Solution of 42 $\int (r^2+r+1)e^r\,dr$

Sol:-

$$\int (r^2+r+1)\,e^r dr = \int r^2 e^r dr + \int r e^r dr + \int e^r dr$$

re^r

diff.		Integ.
r	$+$	e^r
1	$-$	e^r
0		e^r

r^2e^r

diff.		Integ.
r^2	$+$	e^r
$2r$	$-$	e^r
2	$+$	e^r
0		e^r

$$\therefore \int r e^r dr = re^r - e^r$$

$$\int r^2 e^r dr = r^2e^r - 2re^r + 2e^r$$

$$\therefore \int (r^2+r+1)e^r dr = [r^2e^r - 2re^r + 2e^r] + [re^r - e^r] + e^r + c$$

$$= r^2e^r - re^r + 2e^r + c$$

Solution of :- 45 $\int_0^{\frac{\pi}{3}} x \tan^2 x\,dx.$

Sol:-

let $u = x \Rightarrow du = dx$

$dv = \tan^2 x\,dx \Rightarrow v = \int_0 \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$

$= \tan x - x$

$$\therefore \int_0^{\frac{\pi}{3}} x \tan^2 x\,dx = u.v\Big|_0^{\frac{\pi}{3}} - \int_0 v\,du = x\tan x - x^2\Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} (\tan x - x)\,dx$$

$$= \frac{\pi}{3}\tan\frac{\pi}{3} - \frac{\pi^2}{9} - 0 - \int_0^{\frac{\pi}{3}} \tan x\,dx - \int_0^{\frac{\pi}{3}} x\,dx$$

$$= \frac{\pi}{3}\tan\frac{\pi}{3} - \frac{\pi^2}{9} - \ln|\sec x|\Big|_0^{\frac{\pi}{3}} - \frac{x^2}{2}\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3}\cancel{t}\sqrt{3} - \frac{\pi^2}{9} - \ln\left|\sec\frac{\pi}{3}\right| + \ln|\sec 0| - \frac{\pi^2}{18} + 0$$

=

4-29

③ Trigonometric substitutions :-

We shall show how to evaluate integrals that contain expressions of the form :-

$$\sqrt{a^2-x^2}\ ,\quad \sqrt{x^2+a^2}\ ,\quad \sqrt{x^2-a^2}$$

where $a>0$ by making substitutions involving Trigonometric function.

IF the expression of the form $\sqrt{a^2-x^2}$ we set :

$$x = a\sin\theta \ ;\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ . Then:-}$$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta}$$
$$= a\,|\cos\theta| = a\cos\theta$$

since $\cos\theta \geq 0$ since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

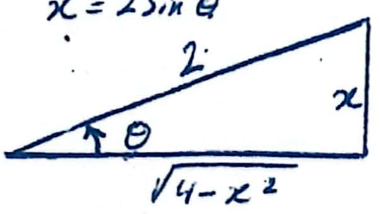
Thus, we have eliminated the radical by this substitution because that we set $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Ex:- Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

$$a = 2 \Rightarrow x = 2\sin\theta \ ;\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$
$$\Rightarrow dx = 2\cos\theta\, d\theta$$

$$\therefore \int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\cdot\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\cdot\sqrt{4(1-\sin^2\theta)}}$$
$$= \int \frac{2\cos\theta\, d\theta}{8\sin^2\theta\cdot\sqrt{\cos^2\theta}} = \int \frac{\cancel{\cos\theta}\, d\theta}{4\sin^2\theta\cdot \cancel{\cos\theta}}$$
$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int \csc^2\theta\, d\theta$$
$$= -\frac{1}{4}\cot\theta + c = -\frac{1}{4}\cot\left(\sin^{-1}\frac{x}{2}\right) + c$$

$x = 2\sin\theta$



ملاحظة :- يمكن استخراج قيمة x بالاعتماد على العلاقات المجاورة في الشكل

$$\cot\theta = \frac{\sqrt{4-x^2}}{x} = \frac{\text{المجاور}}{\text{المقابل}}$$

To eliminate the radical in $\sqrt{x^2+a^2}$ $(a>0)$ we can make the substitution

$$x = a\tan\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} .$$

where :-

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2(\tan^2\theta+1)}$$

$$= a\sqrt{\sec^2\theta} = a\,|\sec\theta| = a\sec\theta$$

$\sec\theta \geq 0$ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

and $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.

Ex:- Evaluate $\int \frac{dx}{\sqrt{x^2+a^2}}$

Sol:-

let $x = a\tan\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

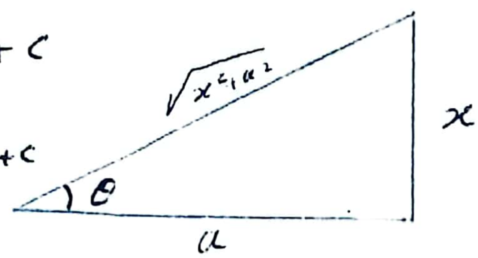
$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{(a^2\tan^2\theta + a^2)}} = \int \frac{a\sec^2\theta\, d\theta}{a\sqrt{\tan^2\theta+1}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C$$

$$= \ln\left|\sqrt{x^2+a^2} + x\right| - \ln|a| + C$$



$$\tan\theta = \frac{\text{المقابل}}{\text{المجاور}} = \frac{x}{a}$$

$$\sec\theta = \frac{\text{الوتر}}{\text{المجاور}} = \frac{\sqrt{x^2+a^2}}{a}$$

4-31

To eliminate the radical $\sqrt{x^2-a^2}$ we can make the substitution

$$x = a\sec\theta \;;\; 0 \le \theta \le \frac{\pi}{2} \text{ or } \pi \le \theta < \frac{3\pi}{2}$$

where:-

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta}$$
$$= a|\tan\theta| = a\tan\theta$$

and

$$\theta = \sec^{-1}\left(\frac{x}{a}\right).$$

Ex:- Evaluate:- $\int \frac{\sqrt{x^2-25}}{x}\,dx$.

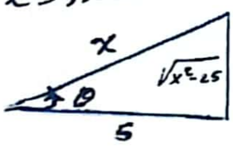
Solution:-

let $x = 5\sec\theta \Rightarrow dx = 5\sec\theta\cdot\tan\theta\,d\theta \;;\; 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$$\therefore \int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{\cancel{5\sec\theta}}\cdot \cancel{5\sec\theta}\tan\theta\,d\theta$$

$$= \int 5\sqrt{\sec^2\theta - 1}\cdot\tan\theta\,d\theta$$
$$= 5\int\sqrt{\tan^2}\cdot\tan\theta\,d\theta = 5\int\tan^2\theta\,d\theta$$

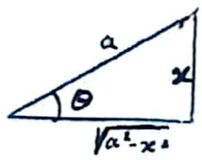
$x = 5\sec\theta$



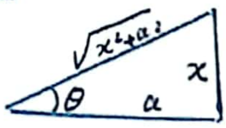
$$= 5\int(\sec^2\theta - 1)\,d\theta = 5\int\sec^2\theta\,d\theta - 5\int d\theta$$
$$= 5\tan\theta - 5\theta + C$$

$$= 5\cdot\frac{\sqrt{x^2-25}}{5} - 5\cdot\sec^{-1}\left(\frac{x}{5}\right) + c = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + c$$

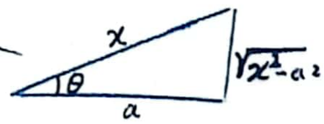
$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$



$\sqrt{x^2+a^2}$ $\quad x = a\tan\theta$



$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$



Exercises :- Evaluate the Integrals -

① $\int \sqrt{4-x^2}\,dx$ ② $\int \sqrt{1-4x^2}\,dx$ ③ $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

④ $\int \frac{dx}{x^2\sqrt{16-x^2}}$ ⑤ $\int \frac{dx}{(4+x^2)^2}$ ⑥ $\int \frac{dx}{(x^2+1)^{3/2}}$

⑦ $\int \frac{\sqrt{x^2-9}}{x}\,dx$ ⑧ $\int \frac{dx}{x^2\sqrt{x^2-16}}$ ⑨ $\int \frac{x^3}{\sqrt{2-x^2}}\,dx$

⑩ $\int x^3\sqrt{5-x^2}\,dx$ ⑪ $\int \frac{dx}{(3+x^2)^{3/2}}$ ⑫ $\int \frac{x^2}{\sqrt{5+x^2}}\,dx$

⑬ $\int \frac{dx}{x\sqrt{4x^2-9}}$ ⑭ $\int \frac{\sqrt{1+t^2}}{t}\,dt$ ⑮ $\int \frac{dx}{(1-x^2)^{3/2}}$

⑯ $\int \frac{dx}{x^2\sqrt{x^2+25}}$ ⑰ $\int \frac{x^2}{1+x^2}\,dx$ ⑱ $\int \frac{dx}{1+2x^2+x^4}$

⑲ $\int \frac{dx}{x^2\sqrt{9-4x^2}}$ ⑳ $\int \frac{x^2}{\sqrt{x^2-25}}\,dx$ ㉑ $\int \frac{dx}{(9x^2-1)^{3/2}}$

㉒ $\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\,d\theta$ ㉓ $\int e^x\sqrt{1-e^{2x}}\,dx$ ㉔ $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

㉕ $\int_0^4 x^3\sqrt{16-x^2}\,dx$ ㉖ $\int_{\sqrt{3}}^{2} \frac{\sqrt{2x^2-4}}{x}\,dx$ ㉗ $\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$

㉘ $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx$ ㉙ $\int_1^3 \frac{dz}{x^4\sqrt{x^2+3}}$ ㉚ $\int_0^3 \frac{x^3}{(3+x^2)^{5/2}}\,dx$

Solution of ⑨ :- $\int \frac{x^3}{\sqrt{2-x^2}}\,dx$

Sol:-

let $x=\sqrt{2}\sin\theta \Rightarrow dx=\sqrt{2}\cos\theta\,d\theta$

$$\therefore \int \frac{x^3}{\sqrt{2-x^2}}\,dx = \int \frac{2\sqrt{2}\sin^3\theta}{\sqrt{2-2\sin^2\theta}}\cdot\sqrt{2}\cos\theta\,d\theta$$

$$= 2\sqrt{2}\int \frac{\sin^3\theta\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}$$

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$$= 2\sqrt{2}\int \frac{\sin^3\theta \cdot \cos\theta \, d\theta}{\sqrt{\cos^2\theta}} = 2\sqrt{2}\int \sin^3\theta \, d\theta$$

$$= 2\sqrt{2}\int (1-\cos^2\theta)\sin\theta \, d\theta$$

$$= 2\sqrt{2}\int \sin\theta \, d\theta - 2\sqrt{2}\int \cos^2\theta \sin\theta \, d\theta$$

$$= -2\sqrt{2}\cos\theta + 2\sqrt{2} \cdot \frac{\cos^3\theta}{3} + C$$

$$= -2\sqrt{2}\cos\theta + \frac{2\sqrt{2}}{3}\cos^3\theta + C$$

$$= -2\cancel{\sqrt{2}}\left(\frac{\sqrt{2-x^2}}{\cancel{\sqrt{2}}}\right) + \frac{2\sqrt{2}}{3}\cdot\left(\frac{\sqrt{2-x^2}}{\sqrt{2}}\right)^3 + C$$

$$= -2\sqrt{2-x^2} + \frac{1}{3}(2-x^2)^{3/2} + C.$$

Solution of (19) $\int \frac{dx}{x^2\sqrt{9-4x^2}}$

Sol:-

let $2x = 3\sin\theta \Rightarrow x = \frac{3}{2}\sin\theta \Rightarrow dx = \frac{3}{2}\cos\theta \, d\theta$

ملاحظة :

يمكن جعل داخل الجذر بالشكل $\sqrt{\frac{9}{4}-x^2}$ واستخراج 2 من داخل الجذر كما :

ثم نعوض كما تعلمنا ونحصل على نفس الناتج لاكمال الحل

$$\therefore \int \frac{dx}{x^2\sqrt{9-4x^2}} = \int \frac{3/2\cos\theta \, d\theta}{\frac{9}{4}\sin^2\theta\sqrt{9-9\sin^2\theta}} = \frac{2}{9}\int \frac{\cos\theta \, d\theta}{\sin^2\theta \cdot \sqrt{1-\sin^2\theta}}$$

$$= \frac{2}{9}\int \frac{\cancel{\cos\theta} \, d\theta}{\sin^2\theta \cdot \cancel{\sqrt{\cos^2\theta}}} = \frac{2}{9}\int \frac{d\theta}{\sin^2\theta} = \frac{2}{9}\int \csc^2\theta \, d\theta$$

$$= \frac{-2}{9}\cot\theta + C$$

$$= \frac{-2}{9}\left[\frac{\sqrt{9-4x^2}}{x}\right] + C$$

$$= \frac{-2\sqrt{9-4x^2}}{9x} + C$$

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5 Integrating Rational Functions; Partial Fractions:-

A rational function $\frac{P(x)}{Q(x)}$ Is called proper if the degree of $P(x)$ is less than the degree of $Q(x)$, otherwise it is improper. thus; It can proved that any proper rational function is expressible as a sum of terms (called partial fractions) having the form:-

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^k}$$

The exact number of termes of each type depends on how the denominator $Q(x)$ factors. In theory, a polynomial $Q(x)$ with real Coefficients can always be factored into a product of linear and quadratic factors with real coefficients.

The quadratic factor which cannot be further decomposed into linear factors without using imaginary numbers. Such quadratic factors are said to be irreducible.

The first step in the partial fraction decomposition of $P(x)/Q(x)$ is to completely factor the denominator $Q(x)$ into linear and irreducible quadratic factors and then collect all repeated factors so that $Q(x)$ is expressed as product of distanct factors of the form

$$(ax+b)^m \quad \text{and} \quad (ax^2+bx+c)^m$$

where ax^2+bx+c is irreducible. Once this is done. the structure of the partial fraction decomposition of $P(x)/Q(x)$ is determined as follows:-

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Linear Factors:-

For each factor of the form $(ax+b)^m$ introduce the m terms:-

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots\cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \dots, A_m$ are constants to be determined.

Irreducible Quadratic Factors:-

For each factor of the form $(ax+bx+c)^m$, introduce m terms:-

$$\frac{Ax_1+B_1}{ax^2+bx+c} + \frac{Ax_2+B_2}{(ax^2+bx+c)^2} + \cdots\cdots + \frac{A_m x+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ are constants to be determined.

Ex:- Evaluate $\int \frac{dx}{x^2+x-2}$

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$= \frac{A(x+2)+B(x-1)}{(x-1)\,(x+2}$$

$$\frac{1}{(x-1)(x+2)} = \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$$

$$= \frac{(A+B)x+(2A-B)}{(x-1)(x+2)}$$

$\therefore$ $A+B=0 \Rightarrow A=-B$

$2A-B=1 \Rightarrow 2(-B)-B=1 \Rightarrow -3B=1$

$\Rightarrow B=-1/3$

$A=-B \Rightarrow A=1/3$.

$$\therefore \frac{1}{x^2+x-2} = \frac{\frac{1}{3}}{x-1} + \frac{-1/3}{x+2}$$

$$\therefore \int \frac{dx}{(x-1)(x+2} = \int \frac{1}{3(x-1)}dx + \int \frac{-1}{3(x+2)}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

Ex:- Evaluate $\int \frac{2x+4}{x^3-2x^2}dx$

Solution:-

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$= \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

$$= \frac{Ax^2-2Ax+Bx-2B+Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2+(-2A+B)x-2B}{x^2(x-2)}$$

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$A + C = 0$ —— ①

$-2A + B = 2$ ---②

$-2B = 4$ ---③ $\Rightarrow \boxed{B = -2}$

from ② and ③ $-2A - 2 = 2 \Rightarrow -2A = 4 \Rightarrow \boxed{A = -2}$

$A + C = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 2$

$$\therefore \frac{2x+4}{x^3-2x^2} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

$$\therefore \int \frac{2x+4}{x^3-2x^2}\,dx = \int \frac{-2}{x}\,dx + \int \frac{-2}{x^2}\,dx + \int \frac{2}{x-2}\,dx$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

$$= -\ln x^2 + \frac{2}{x} + \ln(x-2)^2 + C$$

$$= \ln\left(\frac{(x-2)^2}{x^2}\right) + \frac{2}{x} + C$$

$$= \ln\left(\frac{x-2}{x}\right)^2 + \frac{2}{x} + C$$

ملاحظة :- يمكن حل السؤال بفرضية تجزئة اخرى كالاتي :-

$$\frac{2x+4}{x^3-2x^2} = \frac{(2x+4)}{x^2(x-2)} = \frac{Ax+B}{x^2} + \frac{C}{x-2}$$

التكملة واجب H.W

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Ex:- Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

Sol:-

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{x^2+x-2}{x^2(3x-1)+(3x-1)}$$

$$= \frac{x^2+x-2}{(3x-1)(x^2+1)}$$

$$= \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1)+(Bx+C)(3x-1)}{(3x-1)(x^2+1}$$

$$= \frac{Ax^2+A+3Bx^2-Bx+3Cx-C}{(3x-1)(x^2+1)}$$

$$= \frac{(A+3B)x^2+(-B+3C)x+(A-C)}{(3x-1)(x^2+1)}$$

$\therefore\ x^2+x-2 = (A+3B)x^2+(-B+3C)x+(A-C)$

$\therefore$
$A+3B = 1$ —— ①
$-B+3C = 1$ —— ②
$A-C = -2$ —— ③

from ① $A = 1-3B \Rightarrow C = A+2$
$\Rightarrow C = 1-3B+2$
$\therefore C = 3-3B$

$-B+3C=1 \Rightarrow -B+3(3-3B)=1$
$\Rightarrow -B+9-9B=1 \Rightarrow 9-10B=1$
$\Rightarrow B = \frac{8}{10} \Rightarrow \boxed{B = \frac{4}{5}}$

$\therefore A = 1-3*\frac{4}{5} \Rightarrow A = \boxed{\frac{-7}{5}}$

$C = 3-3B \Rightarrow C = 3-3*\frac{4}{5} \Rightarrow C = 3-\frac{12}{5} \Rightarrow C = \boxed{\frac{3}{5}}$

$$\therefore \int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx = \int \frac{A}{3x-1}\,dx + \int \frac{Bx}{x^2+1}\,dx + \int \frac{C}{x^2+1}\,dx$$

$$= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$= -\frac{7}{5}\ln|3x-1| + \frac{4}{5}*\frac{1}{2}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + c$$

$$= \frac{-7}{5}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

Ex:- Evaluate $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$

Solution:-

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$= \frac{A(x^2+3)^2+(Bx+C)(x+2)(x^2+3)+(Dx+E)(x+2)}{(x+2)(x^2+3)^2}$$

$$= \frac{A(x^4+6x^2+9)+(Bx+C)(x^3+2x^2+3x+6)+(Dx^2+Ex+2Dx+2E)}{(x+2)(x^2+3)^2}$$

$$= \frac{A(x^4+6x^2+9)+(Bx^4+2Bx^3+3Bx^2+6Bx+Cx^3+2Cx^2+3Cx+6C+(Dx^2+Ex+2Dx+2E)}{(x+2)(x^2+3)^2}$$

$$= \frac{(A+B)x^4+(2B+C)x^3+(6A+3B+2C+D)x^2+(6B+3C+E+2D)x+(9A+6C+2E)}{(x+2)(x^2+3)^2}$$

$\therefore$ $A+B=3$

$2B+C=4$

$6A+3B+2C+D=16$

$6B+3C+2D+E=20$

$9A+6C+2E=9$

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ومجموعة المعادلات اعلاه مكونة من خمس معادلات بخمسة متغيرات
تحل بواسطة حل منظومة معادلات خطية.
كما ويمكن حلها باستخدام طريقة التعويض او بطريقة المعادلة :-

$$3x^4+4x^3+16x^2+2x+9$$
$$=A(x^2+3)^2+(Bx+C)(x^2+3)(x+2)+(Dx+E)(x+2)$$

فاذا عوضنا $x=-2$ نحصل على :-

$$49A=49 \Rightarrow A=1$$

وبذلك تكون :-

$$A=1,\ B=2,\ C=0,\ D=4,\ E=0$$

$$\therefore \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}=\frac{1}{x+2}+\frac{2x}{x^2+3}+\frac{4x}{(x^2+3)^2}$$

$$\therefore \int\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}dx=\int\frac{dx}{x+2}+\int\frac{2x}{x^2+3}dx+\int\frac{4x}{(x^2+3)^2}dx$$

$$=\ln|x+2|+\ln(x^2+3)-\frac{2}{x^2+3}+C$$

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Ex:- Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$

$$\begin{array}{r|l} & 3x^2+1 \\ \hline x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\ \text{بالطرح} & \mp 3x^4 \mp 3x^3 \pm 6x^2 \\ \hline & x^2+x-1 \\ & \mp x^2 \mp x \pm 2 \\ \hline & 1 \end{array}$$

$\therefore$ 3

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

$$\therefore \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{dx}{x^2+x-2}$$

$$= x^3+x+\left[\int \frac{x}{(x+2)(x-1)}\right]$$

$$= x^3+x+\left[\int \frac{-1/3}{x+2}\,dx + \int \frac{1/3}{x-1}\,dx\right]$$

$$= x^3+x-\frac{1}{3}\left[\ln|x+2| - \ln|x-1|\right]+c$$

$$= x^3+x-\frac{1}{3}\ln\left|\frac{x+2}{x-1}\right|+c$$

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$= \frac{Ax-A+Bx+2B}{(x+2)(x-1)}$$

$$\Rightarrow A+B=0 \Rightarrow$$

$$-A+2B=1 \Rightarrow$$

$$A=-B \Rightarrow B+2B=1 \Rightarrow B=\frac{1}{3}$$

$$\Rightarrow A=-\frac{1}{3}$$